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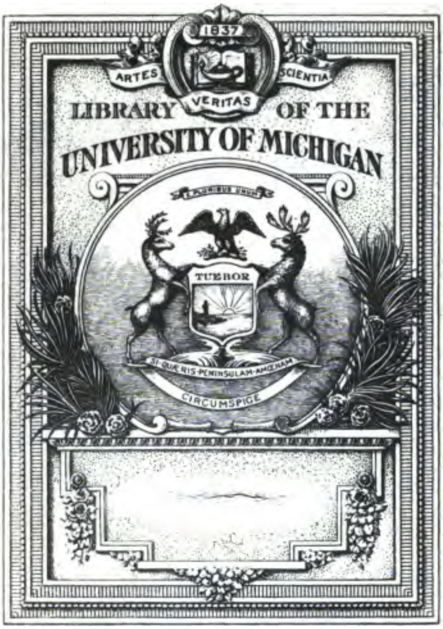
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A

TREATISE

OF

ALGEBRA

In Two BOOKS:

The Firſt treating of the *Arithmetical*, and the Second of the *Geometrical* Part.

By PHILIP RONAYNE, Gent.

The thing that hath been, it is that which ſhall be; and that which is done, that which ſhall be done: And there is no new thing under the Sun. Eccleſ. 1. *ver.* 9.

LONDON:
Printed for *W. Innys* at the *Prince's Arms* in St.
Paul's Church-Yard. 1717.

James M Garraghe

His Book March 24th 1817

TO

His Excellency *Alan* Lord *Brodrick*, Baron of *Middleton*, one of the Lords Juſtices, and Lord High-Chancellor of the Kingdom of *Ireland*.

May it pleaſe Your Excellency,

THE following *Treatiſe* might juſtly be intitled to your Patronage, were it equal to the Extent of the Science treated of, which exceeds all others in Certainty and Clearneſs, and, by a Natural and Regular Method, conduces much to the opening and enlarging of the Mind.

But, tho' it can't pretend to any ſuch Perfection, yet, as it may, in ſome Meaſure, advance ſo uſeful a part of Learning, I take the Liberty to put it under Your Protection, My Lord, who gave early Proofs of a clear Underſtanding, an exact Judgment, and a vaſt Genius capable of comprehending and perfecting the moſt abſtruſe Sciences.

As theſe Qualities made you eminent in your Profeſſion, ſo your Integrity gain'd you ſuch an Univerſal Eſteem, that, with the general Approbation of the People, you were ſoon called into ſeveral Great Stations, and at length deſervedly advanced to the higheſt Dignities in that Kingdom, the Publick Welfare whereof you have conſtantly promoted in all the Employments and Offices which you have ſo honourably filled; And that you may long live the great Ornament thereof is the hearty Wiſh of,

My Lord,

Your Excellency's

Moſt Humble, and

Moſt Obedient Servant

PHILIP RONAYNE.

Hist. Sci.
Bowes
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A
TREATISE
OF
Algebra.

BOOK I.

Definition.

THE Name *Algebra*, Dr. *Wallis* acquaints us, is derived from the firſt Word of *Algiabr*, *Walmokabala*, which Words in the *Arabick* Tongue, ſignifie the *Art* of *Reſtitution* and *Compariſon*, or the *Art* of *Reſolution* and *Æquation*. It is a Science by which the moſt difficult and abſtruſe Queſtions, in *Arithmetick* and *Geometry*, are Reſolved and Demonſtrated; that is, it equally interferes with them both, and therefore it is promiſcuouſly Nam'd, being ſometimes call'd *Specious Arithmetick*, as by *Vieta*, *Harriot*, Dr. *Wallis*, &c. and ſometimes *Modern Geometry*, as by Dr. *Halley*.

This great *Art* may be Defined, or rather deſcribed to be an *Analytical* way of *Demonſtration*; where, aſſuming the Quantity, or Quantities unknown, as if it, or they were known, we proceed by Conſequences, in comparing it or them, and known, or given Quantities among themſelves, untill the unknown Quantity ſought, or at leaſt ſome Power or Powers thereof, be found equal to ſome known Quantities, and ſo it ſelf is, or may be, of Conſequence made known.

CHAP. I.

Concerning the Method *of* Noting *down* Quantities; *and* Tracing *their* Steps, *&c.*

Sect. 1. Notation.

THE Quantity ſought is called the Root, which being unknown, cannot be really expreſs'd; but may be deſign'd by any Symbol, or Character at Pleaſure. I commonly (with moſt others) uſe Vowels for unknown, and Conſonants for known or given Quantities. But *Des Cartes* and his Followers, and moſt Foreign Writers, uſe the laſt Letters of the Alphabet, *x*, *y*, *z* for unknown Quantities; and the firſt Letters, *a*, *b*, *c*, *&c.* for known ones. The advantage of thus expreſſing known Quantities is, that the Numbers with their ſeveral Operations, which in Vulgar Arithmetick would be loſt or ſwallowed up, in *Algebraic* Arithmetick are ſo ordered and managed, as to be preſerved Diſtinct and in View; and at laſt to produce a Canon or Rule for Reſolving, not only the particular Queſtion propos'd, but in general, any other of the like Nature.

Beſides theſe Letters, there are certain Signs uſed, which are as follows.

An Explanation of the Signs uſed in Algebra.

+	Signifies	More, or added to.
−		Leſs, or Subſtracted from.
×		Multiplied by, or Multiplying.
÷		Divided by, or Dividing.
÷)		Continually Divided by.
=		Equal to.
∺		Continual Geometrical Proportion.
∷		Disjunct Geometrical Proportion.
÷		Continual Arithmetical Progreſſion.

Disjunct

∵	Signifies	Disjunct Arithmetical Progreſſion.
⌝ or ⊏		Greater than.
∠ or ⊐		Leſs than.
∽		The Difference of two Quantities, when 'tis not known which of them is the Greater.
●		Involution.
ш		Evolution, or Extracting of Roots.
√		Irrationality, or the Sign of a Surd-Root.

Therefore

$b + c$	Signifies	*b* More *c*, or *c* added to *b*.
$b - c$		*b* Leſs *c*, or *c* Subſtracted from *b*.
$b \times c$ or bc		*b* Multiplyed by *c*, or *b* Multiplying *c*.
$b \div c$ or $\frac{b}{c}$		*b* Divided by *c*, or *c* Dividing *b*.
$b = c$		*b* Equal to *c*.
$a . b : c . d$ ∺		As *a* is to *b*, ſo is *b* to *c*, ſo is *c* to *d*.
$a \cdot\cdot b :: c \cdot\cdot d$		As *a* is to *b*, ſo is *c* to *d*.
$a . b . c . d$ ∺		*a*, *b*, *c*, *d*, have equal Differences.
$a \cdot\cdot b$ ∺ $c \cdot\cdot d$		The Difference of *a* and *b*, is equal to that of *c* and *d*.
b ⌝ *c* or *b* ⊏ *c*		*b* Is Greater than *c*.
b ∠ *c* or *b* ⊐ *c*		*b* Is Leſs than *c*.
b ∽ *c*		The Difference between *b* and *c*, when it is not known which of them is the Greater.
b ●		*b* Is to be involved, or raiſed to ſome Power.
b ш		*b* Is to be Evolved, or ſome Root to be Extracted out of it.

$\sqrt{b}$, (or $\sqrt[2]{b}$), $\sqrt[3]{b}$, $\sqrt[4]{b}$, &c. Signify the Square-Root of *b*, the Cube-Root of *b*, the Biquadrat-Root of *b*, &c. reſpectively.

Beſides the foregoing Signs, (which are commonly uſ'd) I make uſe of others, which are not Common, and are as follows.

$b - \!\ominus$	Signifies,	Some Quantity indefinitely Leſs than b, is to be added to b.
$b\ \ominus\! -$		Some Quantity indefinitely Leſs than b, is to be Subſtracted from b.
$\overset{\mid}{\ominus}$		Either $-\!\ominus$ or $\ominus\!-$, when not material which of them it is.

There are other Signs which are to be uſ'd in the Geometrical Part of Algebra, *and which I will explain in Book II.*

All Quantities concerned in any Queſtion or Problem may ſtand in any Order at pleaſure, *viz.* The moſt convenient for Operation; as $a + b - d$, may ſtand thus, $b - d + a$, or thus, $a - d + b$, or thus, $- d + a + b$, &c; theſe ſtill being the ſame; tho' differently plac'd.

That Quantity which hath no Sign before it (as generally the leading Quantity hath not) is always underſtood to have the Sign $+$ before it, as a is $+ a$, or $b - d$ is $+ b - d$, &c. For the Sign $+$ is an affirmative Sign, and therefore all leading, or Poſitive Quantities are underſtood to have it, as well as they that are to be added.

But the Sign $-$ being a negative Sign, or Sign of Defect, there is a neceſſity of prefixing it to that Quantity to which it belongs, whereever the Quantity ſtands.

When any Quantity is taken more than once, you muſt prefix its Number to it, as $3a$ ſtands for three times a, and $7b$ ſtands for ſeven times b, &c.

All Numbers thus prefixt to any Quantities, are call'd Coefficients, or Fellow-Factors; becauſe they Multiply the Quantity; and if any Quantity be without a Coefficient, it is always ſuppoſed or underſtood to have an Unit prefixt to it; as a is $1\,a$, or $b\,c$, is $1\,b\,c$, &c.

All Quantities that are expreſs'd in Numbers only, (as in Vulgar Arithmetick) are called abſolute Numbers.

Thoſe Quantities that are repreſented by ſingle Letters, as a, b, c, d, &c. Or by ſeveral Letters that are immediately join'd together, as ab, cd, or $7bd$, &c. are call'd Simple, or ſingle whole Quantities.

But when different Quantities repreſented by different or unlike Letters are connected together by the Signs $+$ or $-$, as a

$+ b$, or $a - b$, or $ab + dc$, or $a + aa$. They are call'd Compound whole Quantities.

And when Quantities are expreſs'd, or ſet down like Vulgar Fractions, thus $\frac{a}{b}$ or $\frac{a + d}{b}$, or $\frac{ab + dc}{c - p}$, &c. They are called Fractional or broken Quantities.

Like Quantities are thoſe which are expreſs'd by the ſame Letters under the ſame Power; as b and b, aa and aa, cdb and $6cdb$, &c.

Unlike Quantities are ſuch as are expreſs'd by different Letters, or by the ſame Letters under different Powers, as a and b, cdf and cd, b^3 and b^5 &c,

Sect. 2. Of Tracing the Steps uſed in bringing Quantities to an Æquation.

The Method of tracing the Steps uſed in bringing the Quantities concern'd in any Queſtion to an Æquation, is beſt perform'd by Regiſtring the ſeveral Operations with Figures and Signs placed in the Margent of the Work, according as the ſeveral Operations require; being very uſeful in long and tedious Operations.

For Inſtance, if it be required to ſet down and Regiſter the Sum of the two Quantities a and b, the Work will ſtand

Thus,	1	a
	2	b
$1 + 2$	3	$a + b$

Firſt ſet down the propos'd Quantities a and b, over againſt the Figures 1 and 2, in the ſmall Column (which are call'd Steps) and againſt 3, (the third Step) ſet down the Sum, viz. $a + b$; then againſt the third Step, ſet down $1 + 2$, in the Margin; which denotes that the Quantities againſt the firſt and ſecond Steps are added together, and that thoſe in the third Step are their Sum.

To illuſtrate this in Numbers, ſuppoſe $a = 9$, and $b = 6$, then it will be

Thus,	1	$a = 9$
	2	$b = 6$
$1 + 2$	3	$a + b = 9 + 6 = 15$

Again, if it were required to ſet down the Difference of the ſame two Quantities, Then it will be

Thus,	1	$a = 9$
	2	$b = 6$
$1 - 2$	3	$a - b = 9 - 6 = 3$

Or if it were required to ſet down their Product, then it will be

Thus,	1	$a = 9$
	2	$b = 6$
1 × 2	3	$a \times b$ or $ab = 9 \times 6 = 54$

&c.

Note, Letters Set, or join'd immediately together, (like a Word) ſignifie the Rectangle, or Product of thoſe Quantities they Repreſent; as in the laſt Example, wherein $ab = 54$, is the Product of $a = 9$, and $b = 6$.

Axioms.

1. If equal Quantities be added to equal Quantities, the Sums of thoſe Quantities will be equal,

As if a be $= b$, and $c = d$, then $a + c$ will be $= b + d$, or $a + d = b + c$, by the firſt *Axiom*.

2. If equal Quantities be taken from equal Quantities, the Quantities remaining will be equal; ſo if a be $= b$, and $c = d$, then $a - c$, will be $= b - d$, or $a - d = b - c$, by the ſecond *Axiom*.

3. If equal Quantities be Multiplied by equal Quantities, the Products will be equal. So if a be $= b$, and $c = d$, then $a\,c$ will be $= b\,d$, or $a\,d = b\,c$, by the third *Axiom*.

4. If equal Quantities be Divided by equal Quantities, their Quotients will be equal.

Example, If a be $= b$, and $c = d$, then $\frac{a}{c}$ (or $a \div c$) will be $= \frac{b}{d}$ (or $b \div d$;) or $\frac{a}{d} = \frac{b}{c}$, by the fourth *Axiom*.

5. Thoſe Quantities that are equal to one and the ſame thing, are equal to one another. As for Inſtance, if a be $= q$, and $b = q$, then a will be $= b$, by the fifth *Axiom*.

PART

PART I.

Of whole Quantities.

CHAP. II.

Addition of whole Quantities.

ADdition in *Algebra* may be eaſily Learn'd, by obſerving the following particular *Rules* or *Caſes*.

Rule 1. When Simple and like Quantities having like Signs are to be added,

Add the Coefficients, or prefixt Numbers together, and to their Sum adjoin the Letters common to each, or in either of the ſaid Quantities. Laſtly, to this Sum prefix the common Sign, and you'll have the Sum required.

N.B. Ex. ſignifies *Example.*

		Ex. 1.	*Ex.* 2.	*Ex.* 3.	*Ex.* 4.
	1	*b*	— *a*	20 *bca*	— 52 *bc*
	2	*b*	— *a*	34 *bca*	— 29 *bc*
1 + 2	3	2*b*	— 2*a*	54 *bca*	— 81 *bc*

		Ex. 5.	*Ex.* 6.
	1	+ 654 *abcd*	— 92 *zy*
	2	+ 382 *abcd*	— 24 *zy*
	3	+ 21 *abcd*	— 10 *zy*
	4	+ 1349 *abcd*	— 58 *zy*
1 + 2 + 3 + 4	5	2406 *abcd*	— 184 *zy*

The Reaſon of theſe Additions is evident from the Work of Common Arithmetick; for ſuppoſe b to repreſent 1 *Crown, to which if I add* 1 *Crown, the Sum will be* 2 *Crowns, or* 2 *b, as in* Ex. 1.

Or if we ſuppoſe — a to repreſent the Want, or Debt of one Crown, to which if another Want, or Debt of 1 *Crown be added, the Sum muſt needs be the Want or Debt of* 2 *Crowns, as in* Ex. 2. *And ſo for all the reſt.*

Rule 2. When Simple and like Quantities having unlike Signs are to be added,

Add all the Affirmative ones into one Sum, and all the Negative ones into another, (by the firſt *Rule*) then prefix the Difference of the Coefficients of theſe two Sums, with the Sign of the Greater, to the Letters common to each of the ſaid Quantities, and you'll have the Sum required.

		Ex. 7.	Ex. 8.	Ex. 9.	Ex. 10.
	1	$+5a$	$-5a$	$+bc$	$-9abd$
	2	$-3a$	$+3a$	$-bc$	$+7abd$
1+2	3	$+2a$	$-2a$	$=0bc$, or 0	$-2abd$

		Ex. 11.	
	1	$+9bcdx$	$\}=+16bcdx$
	2	$+7bcdx$	
	3	$-12bcdx$	$\}=-18bcdx$
	4	$-6bcdx$	
1+2+3+4	5	....	$-2bcdx$

The Reaſon of this *Rule* is this,

All Quantities having negative Signs are in Nature directly contrary to ſuch as have Affirmative Signs; and therefore will always deſtroy one another. Thus, if a Man have 1500 l. *in Caſh, and run in Debt* 500 l; *that is, if to his Caſh he add* — 500 l. (*which is the proper way to expreſs a Debt*) *there will remain but* 1000 l. *for the Debt or* — 500 l. *will deſtroy* 500 l *of the Caſh. So alſo if a Man owe* 100 l. *and have nothing to pay it, then he hath* — 100 l. *or is* 100 l. *worſe than Nothing; and if any one give him* 100 l. *or add* + 100 l. *to his* — 100 l. *the Sum will be nothing; but notwithſtanding the the Man (tho' worth Nothing) will be* 100 l. *better than he was before.*

Rule 3. When unlike Quantities are to be added,

Set them all down without altering their Signs; and hence will ariſe Compound Quantities: for unlike Quantities cannot be otherwiſe added but by their Signs.

Thus,		Ex. 12.	Ex. 13.	Ex. 14.
	1	$3a$	$-745cd$	$20b$
	2	$2b$	$-934ca$	$-30c$
1+2	3	$3a+2b$	$-745cd-934ca$	$20b-30c$

		Ex. 15.
	1	$-20bc$
	2	$+40b$
	3	$-30c$
	4	$-59bd$
1+2+3+4	5	$-20bc+40b-30c-59bd$

Rule 4. When compound Quantities are given to be added, find the Sums of the like Quantities, by the firſt and ſecond *Rules*, and then add theſe Sums and the unlike Quantities together, by the third *Rule*, and you'll have the Sum required.

		Ex. 14.
	1	$3aa + 4bcd$
	2	$-2aa - 9bcd + 81e$
	3	$+9aa - 15bcd - 64e + f$
	4	$+7aa + 20bcd \quad - g$
$1 + 2 + 3 + 4$	5	$17aa \; . \; . \; . + 17e + f - g$

CHAP. III.

Subſtraction of whole Quantities.

Subſtraction of whole Quantities is perform'd by one general *Rule*.

Rule.

Change all the Signs of the Subſtrahend; *viz.* of thoſe Quantities which are to be Subſtracted, or ſuppoſe them to be chang'd; then add all the Quantities together, as before in Addition, and their Sum will be the Remainder or Difference required.

That to add — is the ſame thing as to Subſtract +, has been prov'd in Addition; but this general Rule of Subſtraction ſuppoſes that to Subſtract — is all one, as to add +, which Suppoſition may be thus explain'd and prov'd.

If a Man owes 10 l. *more than he is worth, then his Subſtance may (by what has been ſaid in Addition) be repreſented by* — 10 l. *and if any one will pay that* 10 l. *for him, or, which is all one, take away the Debt of* 10 l. *or Subſtract* — 10 l. *he does him as much Service as if he added* 10 l. *to his Caſh.*

		Ex. 1.	Ex. 2.	Ex. 3.	Ex. 4.
	1	$2a$	$-2a$	$8b$	$-15bc$
	2	a	$-a$	$-6b$	$+8bc$
$1 - 2$	3	a	$-a$	$14b$	$-23bc$

		Ex. 5.	Ex. 6.	Ex. 7.
	1	$9abc - 7acd$	$7bc - 2bd + 4$	$bc + 2$
	2	$-6abc - 7acd$	$-bc - 9bd$	$-7bc + 9$
$1 - 2$	3	$15abc$	$8bc + 7bd + 4$	$+8bc - 7$

		Ex. 8.	*Ex.* 9.	*Ex.* 10.
	1	$a+b$	$a+b-54$	0
	2	$a-b$	$-3b-75-bc+d$	$2a-4b$
1 — 2	3	$+2b$	$a+4b+21+bc-d$	$-2a+4b$

		Ex. 11.	*Ex.* 12.
	1	$a+20-97c$	76
	2	$45+d$	$a-b-5d+7c$
1 — 2	3	$a-25-97c-d$	$76-a+b+5d-7c$

		Ex. 13.
	1	b
	2	$-c+d$
1 — 2	3	$b+c-d$

That $-c+d$, *taken from* b, *leaves* $b+c-d$, *for the Remainder, as in the last* Example, *may be thus prov'd.*

Suppoſe	1	$-c+d=a$
	2	Remainder $=r$
Then by the Nature of Subſtraction.	3	$b-a=r$
$3+a$	4	$b-a+a=$ (by what has been ſaid in Addition) b is (by *Axiom* 1.) $=r+a$
By Subſtituting $-c+d$ for a in the latter part of Step 4.	5	$b=r-c+d$
$5+c-d$	6	$b+c-d=r$. Q. E. D.

The Truth of all Operations in Subſtraction, where any doubt ariſes; may be prov'd by adding the Subſtrahend to the Remainder, as in common Arithmetick.

Examples.

	1	$+5a$	0	$-9bc$	
	2	$-2a$	$+3b$	$-6da$	*Subſtrahend.*
1 — 2	3	$+7a$	$-3b$	$-9bc+6da$	*Remainder.*
2 + 3	4	$5a$	0	$-9bc$	*Proof.*

CHAP.

CHAP. IV.

Multiplication of whole Quantities.

Multiplication of whole Quantities admits of Three Caſes.

Caſe 1. If two ſimple Quantities, whether like or unlike, but having like Signs, are to be Multiplied together.

Firſt, Multiply the Coefficients one into another, and then to the Product annex the Letters of both Quantities; ſo ſhall this new Quantity (the Sign + being underſtood as prefixt before it) be the true Product.

		Ex. 1.	*Ex.* 2.	*Ex.* 3.	*Ex.* 4	*Ex.* 5.
	1	*a*	— *a*	— *abcd*	5*d*	— 6*d*
	2	*b*	— *b*	— *fgh*	3*b*	— 7*bc*
1 × 2	3	*ab*	*ab*	*abcdfgh*	15 *db*	42 *dbc*

		Ex. 6.
	1	— 764 *pqr*
	2	— 20 *cl*
1 × 2	3	15280 *pqrcl*

Caſe 2. When the Quantities to be Multiplied are Simple and have unlike Signs.

Join them and the Product of their Coefficients together, as before. But prefix the Sign — before them.

		Ex. 7.	*Ex.* 8.	*Ex.* 9.
	1	*a*	— 6*d*	67916 *plcd*
	2	— *b*	+ 7*b*	— 100 *dq*
1 × 2	3	— *ab*	— 42 *db*	— 6791600 *plcddq*

That in Algebraick Multiplication like Signs muſt give a Poſitive or Affirmative Product, and unlike Signs a Negative one, may be thus prov'd.

I. *Since Multiplication is only adding one Factor (or the Multiplicand) to it ſelf, as often as there are Units in the other (or the Multiplicator); therefore + Multiplying + muſt produce +. Since the Sum ariſing from the Addition of poſitive Quantities muſt be poſitive.*

II. *A Quantity with an affirmative Sign, Multiplying one that hath a negative one, muſt produce a negative Product; for 'tis only adding*

adding the negative Factor to it ſelf, as often as there are Units in ~~the other. Now never ſo many Negatives added together, will ſtill~~ be Negative; and ſo the Product muſt have a negative Sign.

III. *Negative Quantities, Multiplying poſitive ones, muſt give a negative Product; becauſe in this Caſe, the Multiplicator, having a negative Sign, works on the Multiplicand by Subſtraction; which therefore muſt be Subſtracted, or made Negative (by changing its Sign) as often as there are negative Units in the Multiplyer.*

IV. *Negatives Multiplying Negatives, muſt produce an Affirmative, or poſitive Product; becauſe Multiplication by a Negative Quantity, being only a Subſtraction or changing the Sign of the Multiplicand as often as there are negative Units in the Multiplicator; and ſince Subſtracting — is the ſame as adding +, (as was ſhewed in Subſtraction) the defect of the Multiplicand is by this means taken away, and conſequently the Product will be Affirmative.*

Caſe 3. If the Multiplyer and Multiplicand, or either of them, be Compound Quantities; then every Term of the Multiplyer muſt be Multiplyed into all the Terms of the Multiplicand. And the Sum of thoſe particular Products will be the Product required. *As in common Arithmetick.*

Examples.

(1)	1	$a + b - d$
	2	$a - b$
$1 \times a$	3	$aa + ab - ad$
$1 \times -b$	4	$- ab - bb + bd$
$3 + 4$	5	$aa \pm 0 - ad - bb + bd$

(2)	1	$7b + 5d$
	2	$3a - 5f$
$1 \times 3a$	3	$21ab + 15ad$
$1 \times - 5f$	4	$- 35fb - 25fd$
$3 + 4$	5	$21ab + 15ad - 35fb - 25fd$

(3)	1	$aa - ba$
	2	$a + b$
$1 \times a$	3	$aaa - baa$
$1 \times b$	4	$+ baa - bba$
$3 + 4$	5	$aaa - bba$

$1 \times a$

	1	$a + e$
	2	$a - e$
$1 \times a$	3	$aa + ae$
$2 \times -e$	4	$-ae - ee$
$3 + 4$	5	$aa - ee$

That a positive Quantity, Multiplyed by a Negative; or a negative Quantity, Multiplyed by a positive, produces a Negative may be thus prov'd.

If $+ b$, be Multiplied by $- c$, I ſay the Product $= p$, is $= - bc$.

Demonſtration.

ſuppoſe	1	$b = - c + a$	
$1 \times b$	2	$bb = + b \times - c + ba$	$\left\{\right.$ by the Nature of *Multiplication.*
that is	3	$bb = p + ba$	
$1 + c$	4	$b + c = a$	
$4 \times b$	5	$bb + bc = ba$	
$5 - bc$	6	$bb = - bc + ba$	
$3 = 6$	7	$p + ba = - bc + ba$; that is $p = - bc$. Q.E.D.	

That a Negative Multiplyed by a Negative Quantity, produces a Poſitive one, may be prov'd thus.

If $- b$, be Multiplyed by $- c$; I ſay the Product p is $= + bc$.

Demonſtration,

ſuppoſe	1	$a - b = c$	
$1 \times - c$	2	$- ca + \overline{- c \times - b} = - cc$	$\left\{\right.$ by the above *Demonſtration.*
that is	3	$- ca + p = - cc$	
$1 + b$	4	$a = b + c$	
$4 \times - c$	5	$- ca = - cb - cc$	
$5 + cb$	6	$- ca + cb = - cc$	
$3 = 6$	7	$p = cb$. Q. E. D.	

Note, that ſometimes Products are expreſs'd only by the Quantities to be Mulqiplyed with the Sign × between them, thus the Product of a and b, is $a \times b$, and the Product of $a + x$, and $e - z$ is $= \overline{a + x} \times \overline{e - z}$, and the Product of $a + x$, $m - n + y$, and $e + zz$ is $\overline{a + x} \times \overline{m - n + y} \times \overline{e + zz}$.

That $a \times b$ is $= b \times a$, may be thus prov'd.

$$\begin{array}{cccccccc}
1, & 1, & 1, & 1, & 1, & 1, & \&c. & = a\\
1, & 1, & 1, & 1, & 1, & 1, & \&c. & = a\\
1, & 1, & 1, & 1, & 1, & 1, & \&c. & = a\\
1, & 1, & 1, & 1, & 1, & 1, & \&c. & = a\\
\&c. & \&c. & \&c. & \&c. & \&c. & \&c. & & \\
\| & \| & \| & \| & \| & \| & & \\
b & b & b & b & b & b & &
\end{array}$$

Let a conſiſt of any number of Units, as 1, 1, 1, 1, 1, 1, &c. and b be equal to any other number of Units, as 1, 1, 1, 1, &c. Now if you ſuppoſe the number of Units equivalent to a, written over one another, as often as there are Units in b, (as you ſee in this Scheme to the Right Hand) their Sum will be (by the Definition of Multiplication) $= a \times b$.

$$\begin{array}{cccccc}
1, & 1, & 1, & 1, & \&c. & = b\\
1, & 1, & 1, & 1, & \&c. & = b\\
1, & 1, & 1, & 1, & \&c. & = b\\
1, & 1, & 1, & 1, & \&c. & = b\\
1, & 1, & 1, & 1, & \&c. & = b\\
1, & 1, & 1, & 1, & \&c. & = b\\
\&c. & \&c. & \&c. & \&c. & & \\
\| & \| & \| & \| & & \\
a & a & a & a & &
\end{array}$$

And if you ſuppoſe the number of Units equal to b written one over the other, as often as there are Units in a, (as you ſee in this Scheme to the Left Hand) their Sum will be $= b \times a$. But the number of Units in the one Scheme, muſt be equal to the number of Units in the other; altho' a and b were equal to any two Numbers you pleaſe, as is manifeſt, by viewing each Scheme. And conſequently the Sum of the one, or $a \times b$, is $=$ to the Sum of the other, or $b \times a$.

See this Propoſition otherwiſe Demonſtrated in the 16th *Propoſition of the* 7th *Book of* Euclid.

CHAP. V.

Diviſion of whole Quantities.

Diviſion in ſpecies is the converſe, or direct contrary to that of Multiplication, and conſequently perform'd by converſe Operations (as in common Arithmetick.)

The general Rule of Diviſion is this.

Rule.

Place the Diviſor under the Dividend, with a Line between them (as in Vulgar Arithmetick.)

Or place this Sign $\div$ between the Dividend and Diviſor, and let the former be to the Left-Hand of it.

Thus a being Divided by b, will give $\frac{a}{b}$, or $a \div b$, for a Quotient. And $2bcg + dg$, Divided by $6gf - 39a$, is $=$

† 2bcg

$\frac{2bcg + dg}{6fg - 39a}$, or $= \overline{2bcg + dg} \div \overline{6fg - 39a}$.

But if any Quantity be found to be a common Multiplyer in both the Dividend and Diviſor; place that Quantity, which being Multiplyed by that common Multiplyer, will produce the Dividend, over that Quantity which, being Multiplyed by the ſaid common Multiplyer, will produce the Diviſor, with a Line between them.

Examples.

$$\begin{array}{r|c|c|c|c|c}
 & & (3) & (4) & (5) & (6) \\
 & 1 & cfe & abd & 75cd & ab + cdb \\
 & 2 & cb & ab & -5c & a + cd \\
\hline
1 \div 2 & 3 & \frac{fe}{b} & \frac{d}{1} = d & -15d & b
\end{array}$$

$$\begin{array}{r|c|l}
 & & (7) \\
 & 1 & 3ab + 15be - 21db \\
 & 2 & -3b \\
\hline
1 \div 2 & 3 & -a - 5c + 7d
\end{array}$$

Note, when the Quantities in the Diviſor and Dividend are all the ſame, and have the ſame Signs, the Quotient will be an Unit or 1.

Examples.

8. $b \div b = 1$

9. $\frac{a + b}{a + b} = 1$

10. $\frac{a - b}{a - b} = 1$

N. B. *In Diviſion of whole Quantities one thing muſt be carefully obſerved, viz. That like Signs give* +, *and unlike Signs give* —, *which holds here as well as in Multiplication; as may be proved thus,*

Every Dividend being a Product made by Multiplying the Quotient into the Diviſor, the Sign of each Factor muſt be ſuch, as according to the former Rules in Multiplication, can produce the Dividend; wherefore, if the Dividend be Poſitive, the Diviſor and Quotient muſt have like Signs, that is, if the Diviſor be Negative, the Quotient muſt be ſo too, and of conſequence a poſitive Dividend, divided by a negative Diviſor, gives a negative Quotient; and if the Dividend be Negative, the Diviſor and Quotient muſt have unlike Signs; (as has *been Demonſtrated in Multiplication) and conſequently a negative Dividend, divided by a poſitive Diviſor, gives a negative Quotient; but a negative Dividend, divided by a negative Diviſor, gives an affirmative or poſitive Quotient.*

In Diviſion of compound Quantities, the Terms as well of the Dividend as Diviſor, muſt be plac'd in order; according to the Dimenſions of ſome Letter in both of them; that is to ſay, the firſt, ſecond, third, &c. Terms of each muſt be they which have the firſt, ſecond, third, &c. greateſt power of the ſaid Letter reſpectively.

Then you muſt place ſuch a Quantity in the Quotient, as being Multiplied by the firſt Term of the Diviſor ſhall produce the firſt Term of the Dividend, which Quantity Multiply into the whole Diviſor, then Subſtract the Product from the reſpective Terms of the Dividend; and to the Remainder adjoyn, with their proper Signs, as many more of the next following Terms of the Dividend, as is requiſite, and call this Sum your new Dividend. Then find another Term of the Quotient, ſuch as being Multiplyed by the firſt Term of your Diviſor, will produce the firſt Term of your new Dividend, which Term, when found, place in your Quotient, then Multiply and Subſtract as aforeſaid; and ſo proceed.

Thus, If $aaa + 3aae + 3aee + eee$, be given to be Divided by $a + e$; I place them in this manner

$$\begin{array}{c} aaa + 3aae + 3aee + eee \\ a + e \end{array}$$

then I ſeek ſuch a Quantity as being Multiplyed by a (the firſt Term of the Diviſor) will produce aaa, (the firſt Term of the Dividend) and find it to be aa; and therefore I place aa in the Quotient, then I Multiply aa by $a + e$, and the Product $aaa + aae$, I Subſtract from $aaa + 3aae$, (the reſpective Terms of the Dividend) and the Remainder is $2aae$, to which I adjoin the next Term of the Dividend, viz. $+ 3aee$, and the Sum $2aae + 3aee$ is my new Dividend. Again, I ſeek a Quantity, which being Multiplyed by a, (the firſt Term of the Diviſor) will produce $2aae$, (the firſt Term of the new Dividend) and find it to be $+ 2ae$, which $+ 2ae$, I place in the Quotient, then I Multiply it by (or into) $a + e$, and the Product $2aae + 2aee$, I Subſtract from $2aae + 3aee$, and the Remainder is aee, to which I adjoin the next and laſt Term of the Dividend, viz. $+ eee$, and the Sum $aee + eee$ is my new Dividend.

Then I ſeek a Quantity, which being Multiplyed by a, will produce aee, and finding it to be ee, I place $+ ee$ in the Quotient, and Multiply $a + e$ by it, and the Product $aee + eee$, I Subſtract from my laſt foregoing Dividend, and the Remainder is nothing. See the following Operation.

Ex. 11.

			Quotient
	1	$aaa + 3aae + 3aee + eee$	$(aa + 2ae + ee$
	2	$a + e$	
$2 \times aa$	3	$aaa + aae$	
$aaa + 3aae - 3$	4	$0 + 2aae$	
$4 + 3aee$	5	$2aae + 3aee$	
$2 \times 2ae$	6	$2aae + 2aee$	
$5 - 6$	7	$0 + aee$	
$7 + eee$	8	$aee + eee$	
$2 \times ee$	9	$aee + eee$	
$8 - 9$	10	0	

Or Diviſion of Quantities may be better perform'd thus,

Ex. 12.

$$\begin{array}{l}
3a-6)\quad 6aaaa-96 \quad (2aaa+4aa+8a+16 \\
\quad\quad\quad\ 6aaaa-12aaa \\
\hline
\quad\quad\quad\quad 0 \quad +12aaa \\
\quad\quad\quad\quad\quad\ +12aaa-24aa \\
\hline
\quad\quad\quad\quad\quad\quad 0 \quad +24aa \\
\quad\quad\quad\quad\quad\quad\quad +24aa-48a \\
\hline
\quad\quad\quad\quad\quad\quad\quad\quad 0 \quad +48a-96 \\
\quad\quad\quad\quad\quad\quad\quad\quad\quad\ 48a-96 \\
\hline
\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad 0
\end{array}$$

Suppoſe it was required to divide $aaa + 4aac + daa + dda + 4dac + ddd$, by $a + d$.

If aaa be made the firſt Term of the Dividend, $+ 4caa + daa$, $+ 4dca + dda$, $+ ddd$, muſt be the ſecond, third, and fourth Terms of it reſpectively; and a, $+ d$, the firſt and ſecond Terms of the Diviſor reſpectively; and then the Diviſion will ſtand thus,

$$\begin{array}{rllll}
a+d\,) & aaa & \begin{matrix}+4caa\\+daa\end{matrix} & \begin{matrix}+4cda\\+dda\end{matrix} & +ddd\;\;(\,aa+4ac+dd\\
 & aaa+daa & & & \\ \hline
 & & 0\;\;+4caa & +4cda & \\
 & & \;\;\;\;\;+4caa & +4cda & \\ \hline
 & & & 0\;\;+dda & +ddd\\
 & & & \;\;\;\;\;dda & +ddd\\ \hline
 & & & & 0
\end{array}$$

N.B. The foregoing Dividend may be writ in the following manner,

$$aaa\begin{matrix}+4c\\+d\end{matrix}aa\begin{matrix}+4cd\\+dd\end{matrix}a+ddd$$

In which the Members of the ſecond Term, as alſo they of the third are United, by adding in each Term the Factors of the Letter (a), in reſpect of which the Terms of the Dividend were plac'd.

But if ddd be made the firſt Term of the Dividend, then the Diviſion will ſtand thus,

$$\begin{array}{rlll}
d+a\,) & ddd+add & \begin{matrix}+aa\\+4ca\end{matrix}d & \begin{matrix}+4caa\\+aaa\end{matrix}\;\;(\,dd\begin{matrix}+aa\\+4ca\end{matrix}\\
 & ddd+add & & \\ \hline
 & 0 & \begin{matrix}+aa\\+4ca\end{matrix}d & \begin{matrix}+4caa\\+aaa\end{matrix}\\
 & & \begin{matrix}aa\\+4ca\end{matrix}d & \begin{matrix}+4aac\\+aaa\end{matrix}\\ \hline
 & & & 0
\end{array}$$

Again,

$$\begin{array}{rlll}
{*}\;\begin{matrix}g\\h\end{matrix}a-\begin{matrix}g\\h\end{matrix}b\,) & -aa\begin{matrix}+b\\+c\end{matrix}a-bc & & \left(\dfrac{-a}{g+h}+\dfrac{c}{g+h}=\dfrac{-a+c}{g+h}\right.\\
 & -aa+ba & & \\ \hline
 & 0\;+ca-cb & & \\
 & \;\;\;\;\;\;ca-cb & & \\ \hline
 & \;\;\;\;\;\;\;\;0 & &
\end{array}$$

* That is, $\overline{g+h}\times a-\overline{g+h}\times b$ for the Di=

, and $-aa+ba+ca-bc$ for the Divi=

If the Diviſor be not an Aliquot part of the Dividend, the Quotient may, in ſome Caſes, be continued to an infinite Series. But if, after you have plac'd as many Terms in the Quotient as you think proper, you have a mind to have the exact Quotient, place the laſt Remainder as a Numerator over your Diviſor, which Fraction annex to the before found Quotient, with its proper Sign + or —.

Examples.

$$1 - x \,)\, 1 \,(1 + x + xx + xxx + \text{\&c. Sine Fine.}$$

$$\text{Or } 1 + x + xx + xxx + \frac{xxxx}{1 - x}$$

$$\begin{array}{rrrr}
1 - x & & & \\
\hline
0 + x & & & \\
+ x & - xx & & \\
\hline
0 & + xx & & \\
& + xx & - xxx & \\
\hline
& 0 & + xxx & \\
& & + xxx & - xxxx \\
\hline
& & & + xxxx
\end{array}$$

=4

$1+\frac{3}{4}+\frac{9}{16}+\frac{[illegible]}{64}+\frac{81}{256}$

$1-3 \,|\, 4$

$4-3$

$+3$

$+3 \; - \frac{9}{4}$

$\frac{9}{4} - \frac{27}{16}$

$+\frac{27}{16} - \frac{81}{64}$

$\frac{81}{64}$

$1-,5 \,)\, 1 \,|\, 1+,5+,25+,125$

$1-,5$

$+,5$

$,5-,25$

$+,25$

$,25-,125$

$+,125$

or $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ &c $=2$

$1-,5 = ,5 \,|\, 1.0 \,|\, 2$

$x-1 \,|\, \frac{x}{x-1} \,|\, 1+\frac{1}{x}+\frac{1}{xx}+\frac{1}{xxx}+\frac{1}{xxxx}$ &c

$+1 - \frac{1}{x}$

$+\frac{1}{x}$

$+\frac{1}{x} - \frac{1}{xx}$

$+\frac{1}{xx}$

$+\frac{1}{xx} - \frac{1}{xxx}$

$+\frac{1}{xxx}$

$2-1 \,|\, \frac{2}{2-1} \,|\, 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$

$+1$

$+1 - \frac{1}{2}$

$+\frac{1}{2}$

$+\frac{1}{2} - \frac{1}{4}$

$+\frac{1}{4}$

$+\frac{1}{4} - \frac{1}{8}$

$+\frac{1}{8}$

$+\frac{1}{8} - \frac{1}{16}$

$$aa - ee \,)\, aae \,(e + \frac{eee}{aa} + \frac{eeeee}{aaaa} + \text{\&c.}$$

$$\begin{array}{rrr}
aae - eee & & \\
\hline
+ eee & & \\
eee & - \dfrac{eeeee}{aa} & \\
\hline
& + \dfrac{eeeee}{aa} & \\
& + \dfrac{eeeee}{aa} & - \dfrac{eeeeeee}{aaaa} \\
\hline
\end{array}$$

$$\text{Remainder } \frac{eeeeeee}{aaaa}$$

whence $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$ in inf is equal to 1. vide 3

in like manner $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}$ in infinity $=\frac{1}{2}$ vide 2

& $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}$ &c $=\frac{1}{3}$ vid 3d

& $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\frac{1}{625}$ &c $=\frac{1}{4}$ vid 4th

& by multiplying ye numer. you will find so much value of the fract. as multiplied accordingly thus $\frac{2}{4}+\frac{2}{16}+\frac{2}{64}+\frac{2}{256}=\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}=\frac{2}{3}$ &

$1+\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}$ &c $=3$

$3-2 \,|\, \frac{3}{3-2} \,|\, 3 \,|\,$ & $\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\frac{16}{81}=2$

$+2$

$2 - \frac{4}{3}$ by halving

$+\frac{4}{3} - \frac{8}{9}$ | $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\frac{8}{81}=1$

by halving

$+\frac{8}{9} - \frac{16}{27}$ $\frac{1}{6}+\frac{2}{18}+\frac{4}{54}+\frac{8}{162}=\frac{1}{2}$

$\frac{16}{27}$

2d 4th

2d ,33333 | ,25 | ,2
11111 | 625 | 48
3703 | 156 | 16
1234 | 39 | 32
411 | 9 | 6
137 | 2331 | ,24998 in 5
near to 45 | | or ,25
in 5 15
5
1
,49995

$10-5 \,|\, \frac{10}{10-5} \,|\, 1+\frac{5}{10}+\frac{25}{100}+\frac{125}{1000}$ &c $=2$

$+5$

$+5 - \frac{25}{10}$

$\frac{25}{10} - \frac{125}{100}$

$+\frac{125}{1000}$

By reducing the fractions $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ &c $=2$

1,5
,25
,12
,6
,31
,15
,7
,3
,1

,5
125
,3125
,781
195
48
1.2
3

PART II.

Of Fractional or Broken Quantities.

CHAP. I.

Notation of Fractional Quantities.

FRactional *Quantities* are expreſs'd or ſet down like Vulgar *Fractions*, in common Arithmetick.

$$\text{Thus}\left\{\frac{a}{b},\ \frac{5b-4a}{4d+7b},\ \frac{2bc}{d}\right.\quad\begin{array}{l}\textit{Numerators.}\\ \textit{Denominators.}\end{array}$$

How they came to be ſo, may be ſeen by the general Rule in the beginning of Diviſion.

Theſe *Fractional Quantities* are manag'd in *Algebra*, as Broken Numbers in *Arithmetick*.

CHAP. II.

Reduction of Fractional Quantities.

Sect. 1. To reduce Fractions having different Denominators, to Fractions of the ſame Value, that ſhall have a common Denominator.

Rule.

MUltiply all the Denominators continually by each other, (*i. e.* Multiply the firſt Denominator by the ſecond, and that Product by the third, and ſo on) and reſerve their Product for a new and common Denominator; then Multiply any of the Numerators by all the Denominators but its own, continually, and the laſt Product put for a Numerator over the ſaid common Denominator; *ſo this Fraction is equal to that, whoſe Numerator you Multiplied into all the Denominators, but its own continually.* Do ſo with the reſt of the Numerators, and you'll have your deſire.

Examples.

1. Let it be required to bring $\frac{a}{b}$ and $\frac{d}{c}$ to one Denomination.

Firſt, $b \times c = bc$, is the common Denominator.

Secondly, $a \times c = ac$; therefore $\frac{ac}{bc}$ is $= \frac{a}{b}$.

Thirdly, $b \times d = bd$; therefore $\frac{bd}{bc}$ is $= \frac{d}{c}$.

Conſequently $\frac{ac}{bc}$ and $\frac{bd}{bc}$ are the two Fractions required.

2. Let $\frac{cdf}{e}$, $\frac{ab}{fd}$, and $\frac{a+d}{c-d}$, be all brought to a common Denominator.

Firſt, $e \times fd \times \overline{c-d} = efdc - efdd$, is the common Denominator.

Secondly, $cdf \times fd \times \overline{c-d} = ccddff - cdddff$; $ab \times e \times \overline{c-d} = abec - abed$; and $\overline{a+d} \times e \times fd = aefd + ddef$, are the Numerators,

Therefore $\frac{ccddff - cdddff}{efdc - efdd}$, $\frac{abec - abed}{efdc - efdd}$,

And $\frac{aefd + ddef}{efdc - efdd}$, are the Fractions required.

3. If $\frac{b+c}{a+b}$ and $\frac{d-c}{b-d}$ be brought to one Denomination, they will be $\frac{bb+bc-bd-dc}{ba+bb-da-bd}$ and $\frac{ad-ac+bd-bc}{ba+bb-da-bd}$ reſpectively.

Sect. 2. To reduce a whole Quantity into an Æquivalent Fraction of a given Denomination.

Rule.

Multiply the whole Quantity by the given Denominator, under which Product place the ſaid Denominator with a Line between them, and you will have the Fraction required.

Examples.

1. Let it be required to bring $a+b$ into a Fraction, whoſe Denominator ſhall be $d-a$.

Firſt, $\overline{a+b} \times \overline{d-a} = da + db - aa - ba$,

Then $\frac{da+db-aa-ba}{d-a}$, is the Fraction required.

2. *b* being reduc'd to a Fraction, whoſe Denominator will be $q-r$ is $=\frac{bq-br}{q-r}$.

Note, when whole Quantities are to be ſet down Fraction-wiſe; ſubſcribe an Unit for the Denominator; thus ab *is* $=\frac{ab}{1}$, *and* $aa+b=\frac{aa+b}{1}$.

Lemma to *Sect.* 3.

How to find the greateſt common Diviſor (or common Meaſure) of two given whole Quantities.

Rule.

Divide the Greater given Quantity by the Leſs, and if any thing remains, Divide your Diviſor thereby, and if any thing yet remains, Divide your laſt Diviſor thereby; and thus proceed till nothing remains (*as in Vulgar Fractions*), if the firſt Term of each Remainder be a Sub-Multiple of the firſt Term of the next foregoing Diviſor: But here *Note*, that every Remainder whoſe firſt Term is not a Sub-Multiple of the firſt Term of the next foregoing Diviſor, muſt be Divided by ſuch of its Sub-Multiples as is Prime to that Diviſor, and will give the firſt Term of the Quotient a Sub-Multiple of the firſt Term of that Diviſor: by which Quotient that Diviſor muſt be Divided, and if there be a Remainder, the ſaid Quotient will be your next Dividend to be Divided by this Remainder, or &c. And the laſt Diviſor is the greateſt common one required.

Suppoſe the Greater of the two given Quantities to be $=b$, and the Leſs $=c$; and ſuppoſe c to be * Divided by a Sub-Multiple thereof $=f$ Prime to b, and giving the firſt Term of the Quotient $\frac{c}{f}$ a Sub-Multiple of the firſt Term of b (*where Note, that* f *will be* $=1$, *if the firſt Term of* c *be a Sub-Multiple of the firſt Term of* b); then ſuppoſe b to be Divided by $\frac{c}{f}$, and the Remainder $=r$. Again, ſuppoſe r to be Divided by a Sub Multiple thereof $=g$ Prime to $\frac{c}{f}$, and giving the firſt

* *Note, this Diviſion is not neceſſary nor proper, if the firſt Term of* c *be a Sub-Multiple of the firſt Term of* b.

Term of the Quotient $\frac{r}{g}$ a Sub-Multiple of the firſt Term of $\frac{c}{f}$: then ſuppoſe $\frac{c}{f}$ to be Divided by $\frac{r}{g}$, and the Remainder $= s$. Again, ſuppoſe s to be Divided by a Sub-Multiple thereof $= b$ Prime to $\frac{r}{g}$, and giving the firſt Term of the Quotient $\frac{s}{b}$ a Sub-Multiple of the firſt Term of $\frac{r}{g}$: and then $\frac{r}{g}$ being Divided by $\frac{s}{b}$, ſuppoſe the Remainder $= 0$. I ſay $\frac{s}{b}$ is $=$ the greateſt common Diviſor required.

Demonſtration.

* $\frac{s}{b}$ [a] Meaſures $\frac{r}{g}$, and $\frac{r}{g}$ [a] Meaſures $\frac{c}{f} - s$; therefore $\frac{s}{b}$ [b] Meaſures $\frac{c}{f} - s$: But $\frac{s}{b}$ Meaſures s; therefore $\frac{s}{b}$ [c] Meaſures $\frac{c}{f} - s + s = \frac{c}{f}$; and conſequently c. Again, $\frac{c}{f}$ [a] Meaſures $b - r$: therefore $\frac{s}{b}$ Meaſures $b - r$: But $\frac{s}{b}$ [a] Meaſures $\frac{r}{g}$, and conſequently r; therefore $\frac{s}{b}$ [c] Meaſures $b - r + r = b$. Whence'tis certain that $\frac{s}{b}$ is a common Meaſure to, or of b and c.

Now in order to prove that it is the Greateſt, let w be $=$ the greateſt common Meaſure of b and c. Then f being Prime to b, and w Meaſuring b; f will therefore

* i. e. *The value of* $\frac{s}{b}$

[a] *Suppoſition.*

$\left\{\begin{matrix} b \\ c \end{matrix}\right\}$ *Axiom* $\left\{\begin{matrix} 11 \\ 12 \end{matrix}\right\}$, *the* 7*th Book of* Eucl. El.

fore be [d] Prime to w; and f and w [a] Meaſuring c; wf will [e]Meaſure c, or w will Meaſure $\frac{c}{f}$. But $\frac{c}{f}$ [a] Meaſures $b - r$; therefore w will Meaſure $b - r$, and [c] conſequently $b - \overline{b - r} = r$. Again, g being [a]Prime to $\frac{c}{f}$, and w Meaſuring $\frac{c}{f}$; g will therefore be [d]Prime to w, and [a]g and w Meaſuring r; gw will [e] Meaſure r, or w will Meaſure $\frac{r}{g}$: But $\frac{r}{g}$ Meaſures $\frac{c}{f} - s$; therefore w will Meaſure $\frac{c}{f} - s$; and [c]conſequently $\frac{c}{f} - \overline{\frac{c}{f} - s} = s$. Again, b being [a]Prime to $\frac{r}{g}$, and w Meaſuring $\frac{r}{g}$; b will therefore be [d] Prime to w; and b and w Meaſuring s; bw will [e]Meaſure s; *viz.* w will Meaſure $\frac{s}{b}$; conſequently w cannot exceed $\frac{s}{b}$; therefore they are equal.

[d] 25. 7. Eucl. El.

[e] *a conſequence from the* 36 *and* 37. 7 Eucl. El.

Finally, it is evident, that the like Demonſtration with this may be applied to any other Number of Quantities. *Q. E. D.*

Sect. 3. To abbreviate or bring a Fractional Quantity into its loweſt Denomination.

Rule.

Divide the Numerator and Denominator ſeverally by their greateſt common Diviſor, and the reſpective Quotients plac'd Fraction-wiſe, is the Fraction required.

Examples.

1. Let it be required to reduce $\frac{aac}{cd}$ to its leaſt Terms.

aac being Divided by c (which c is the greateſt common Diviſor of aac and dc), gives aa for a new Numerator; and dc being Divided by c, gives d for a new Denominator: conſequently, $\frac{aa}{d}$ is the Fraction required.

2. In like manner $a + \frac{bde}{bc}$ is found to be $= a + \frac{d}{1} = a + d$.

In ſuch ſingle Fractions as theſe, the common Diviſors (if there be any) are eaſily diſcover'd by inſpection only: But in compound Fractions it muſt (for the moſt Part) be found by our Lemma thus,

3. Let it be required to reduce $\frac{aa + 2ba + bb}{a^3 - bba}$ to its loweſt Terms.

Firſt I Divide the greater of the two given Quantities, *viz.* That which has the Greateſt power of a in't, (in reſpect of which Letter the Terms were plac'd) by the Leſs thus,

$$\begin{array}{rl}
aa + 2ba + bb) & a^3 - bba \quad (a - 2b \\
& \underline{a^3 + 2baa + bba} \\
& \quad -2baa - 2bba \\
& \quad \underline{-2baa - 4bba - 2b^3} \\
& \qquad\qquad + 2bba + 2b^3
\end{array}$$

Now ſince the firſt Term of the Diviſor has a greater Power of a in it than the firſt Term of $2bba + 2b^3$; this therefore is a Remainder: But the firſt Term thereof, *viz.* $2bba$ is not a Sub-Multiple of aa, the firſt Term of the ſaid Diviſor; wherefore I Divide the ſaid Remainder by $2bb$ a Sub-Multiple thereof, Prime to the ſaid Diviſor, and giving the firſt Term of the Quotient a Sub-Multiple of the firſt Term of the aforeſaid Diviſor; thus

$$\begin{array}{rl}
2bb) & 2bba + 2b^3 \quad (a + b \\
& \underline{2bba \qquad\quad} \\
& 0 + 2b^3 \\
& \quad \underline{\ \ 2b^3} \\
& \qquad 0
\end{array}$$

And the Quotient is $a + b$; by which I Divide the ſaid Diviſor; thus

$$\begin{array}{lll}
a+b) & aa+2ba+bb & (a+b \\
 & \underline{aa+ba} & \\
 & \quad ba+bb & \\
 & \quad \underline{ba+bb} &
\end{array}$$

And the Remainder is 0

Whence $a + b$ is the greateſt common Meaſure required, by which the given Fraction will be reduc'd to its loweſt Terms; thus

$$\begin{array}{lll}
a+b) & aa+2ba+bb & (a+b \text{ the new Numerator.} \\
 & \underline{aa+ba} & \\
 & \quad ba+bb & \\
 & \quad \underline{ba+bb} & \\
 & \qquad 0 &
\end{array}$$

$$\begin{array}{lll}
a+b) & a^3-bba & (aa-ba \text{ the new Denominator.} \\
 & \underline{a^3+baa} & \\
 & \quad -baa-bba & \\
 & \quad \underline{-baa-bba} & \\
 & \qquad 0 &
\end{array}$$

Conſequently $\dfrac{a+b}{aa-ba}$ is the Fraction required.

4. In like manner $\dfrac{x^4-b^4}{x^5-x^3bb}$ will be reduc'd to its loweſt Terms thus,

$$\begin{array}{lll}
x^4-b^4) & x^5-x^3bb & (x \\
 & \underline{x^5-xb^4} & \\
 & \quad -x^3bb+xb^4 & \text{Remainder.}
\end{array}$$

$$\begin{array}{lll}
-bb) & -x^3bb+xb^4 & (x^3-xbb) \\
 & \underline{-x^3bb} & \\
 & \quad 0 \;\; +xb^4 & \\
 & \qquad \underline{+xb^4} & \\
 & \qquad 0 &
\end{array}
\qquad
\begin{array}{ll}
x^4-b^4 & (x \\
\underline{x^4-xxbb} & \\
+xxbb-b^4 & \text{Remainder.}
\end{array}$$

$$\begin{array}{rlll}
bb)\ xxbb & -b^4 & (xx-bb)\ x^3-xbb & (x \\
\underline{xxbb} & & \underline{x^3-xbb} & \\
0 & -b^4 & \quad 0 & \textit{Remainder.} \\
 & \underline{-b^4} & & \\
 & \quad 0 & &
\end{array}$$

Hence $xx - bb$ is the greateſt common Diviſor of $x^4 - b^4$ and $x^5 - x^3b^2$, by which $\frac{x^4 - b^4}{x^5 - x^3bb}$ will be reduc'd to $\frac{xx+bb}{x^3}$ by this *Sect.*

5. Let it be required to reduce $\frac{-xx+cx+x-c}{xx-2x+1}$ to its loweſt Terms.

Firſt I Divide $-xx+cx+x-c$ by $xx-2x+1$, thus

$$\begin{array}{rl}
xx-2x+1) & -xx\begin{smallmatrix}+c\\+1\end{smallmatrix}x-c\ (-1 \\
 & \underline{-xx+2x-1} \\
\text{And the Remainder is} & \quad\begin{smallmatrix}+c\\-1\end{smallmatrix}x\begin{smallmatrix}-c\\+1\end{smallmatrix}
\end{array}$$

But the firſt Term of this Remainder; *viz.* $\begin{smallmatrix}+c\\-1\end{smallmatrix}x$ is not a Sub-Multiple of the firſt Term of the foregoing Diviſor; to wit of xx; wherefore I Divide the ſaid Remainder by its Sub-Mulmultiple $\begin{smallmatrix}+c\\-1\end{smallmatrix}$ which is Prime to the ſaid Diviſor, and will give the firſt Term of the Quotient a Sub Multiple of the firſt Term of the ſaid Diviſor; thus,

$$\begin{array}{rll}
\begin{smallmatrix}+c\\-1\end{smallmatrix}) & \begin{smallmatrix}+c\\-1\end{smallmatrix}x & \begin{smallmatrix}-c\\+1\end{smallmatrix}\ (+x-1 \\
 & \underline{\begin{smallmatrix}+c\\-1\end{smallmatrix}x} & \\
 & \quad 0 & \begin{smallmatrix}-c\\+1\end{smallmatrix} \\
 & & \underline{\begin{smallmatrix}-c\\+1\end{smallmatrix}} \\
 & & \quad 0
\end{array}$$

And the Quotient is $+x-1$, by which I Divide the aforeſaid Diviſor thus,

$$\begin{array}{r l}
x-1)\ & xx-2x+1\ (x-1 \\
& \underline{xx-\ \ x\qquad} \\
& \quad -x+1 \\
& \quad \underline{-x+1} \\
\end{array}$$

And the Remainder is 0

Wherefore $x-1$ is the greateſt common Diviſor ſought, by which the given Fraction will be reduc'd to $\frac{-x+c}{x-1}$ by this Sect.

If by proceeding in the foregoing Manner, you find the greateſt common Diviſor to be 1; then the Fraction is in its leaſt Terms already.

Note, *Some Fractional Quantities are reduc'd to their leaſt Terms by Dividing the Numerator and Denominator ſeverally, by the greateſt common Meaſure of their ſeveral Members; thus* $\frac{49ed-42cd+294bd}{35qbd+28dc}$ is reduc'd to $\frac{7e-6c+42b}{5qb+4c}$ *by Dividing the given Numerator and Denominator ſeverally by 7d, the greateſt common Diviſor of their ſeveral Members.*

Sect. 4. To reduce a Compound Fraction, to a Simple one of the ſame Value.

Rule.

Multiply the Numerators by each other continually for a new Numerator, and the Denominators continually for a new Denominator; ſo this new Fraction is that required.

Examples.

1. Let it be required to reduce $\frac{3}{b}$ of $\frac{1}{c}$ of $\frac{a}{d}$ to a Simple Fraction of the ſame Value.

First, $3\times 1\times a=3a$, the new Numerator.
Secondly $b\times c\times d=bcd$, the new Denominator.

Conſequently $\frac{3a}{bcd}$ is the new Fraction required.

2. Let it be required to reduce $\frac{a+b}{f}$ of $\frac{1}{c+d}$ of $\frac{1}{p}$ of $\frac{q}{r}$ to a Single Fraction of the ſame Value.

$a+b$

$\overline{a + b} \times 1 \times 1 \times q = aq + bq$, the new Numerator.

$f \times \overline{c + d} \times p \times r = fcpr + fdpr$, the new Denominator.

Conſequently $\frac{aq + bq}{fcpr + fdpr}$ is the Simple Fraction required.

CHAP. III. and IV.

Addition and Subſtraction of Fractional Quantities.

What hath been done by the Rules in the next foregoing Chapter, is chiefly to fit and prepare Fractions of different Denominations for Addition, or Subſtraction, as occaſion requires; viz.

1. If the Fractions given to be added, or Subſtracted, be Compound ones, they muſt be reduc'd to Simple or Pure Fractions (by *Sect.* 4. of the next foregoing Chap.)

2. If they have not a Common Denominator, they muſt be reduc'd to Fractions of the ſame Value, that will have a Common Denominator (by *Sect.* 1. Chap. 2.)

That being done, *Addition* and *Subſtraction* are thus perform'd.

Rule.

Add or *Subſtract* their Numerators, as occaſion requires; and under their Sum or Difference, ſubſcribe the Common Denominator.

Examples in Addition.

	1	$\frac{a + b}{d}$	$\frac{2a + b}{d + c}$	$\frac{a - b + d}{d + a}$
	2	$\frac{aa}{d}$	$\frac{2b - a}{d + c}$	$\frac{a + b - d}{d + a}$
1 + 2	3	$\frac{a + b + aa}{d}$	$\frac{a + 3b}{d + c}$	$\frac{2a}{d + a}$

Suppoſe it was required to add $b + \frac{c}{d}$, $+ f - \frac{q}{p}$, and $g + \frac{c}{p}$ into one Sum.

Firſt the Fractional Parts are $\frac{c}{d} - \frac{q}{p}$ and $+ \frac{c}{p}$.

Secondly, $d \times p \times p = dp^2$, the common Denominator.

Thirdly, $c \times p \times p = cp^2$.

Fourthly, $-q \times d \times p = -qdp$.

Fifthly, $c \times d \times p = cdp$; Wherefore $\frac{cp^2 - qdp + cdp}{dpp} = \frac{cp - qd + cd}{dp}$ (by *Sect.* 3. Chap. II.) is the Sum of the Fractional Parts: Consequently the Sum required is $b + f + g + \frac{cp - qd + cd}{dp}$.

Again, let it be required to add $\frac{1}{b}$ of $\frac{d}{p}$ to $\frac{d-q}{bq}$.

First, $1 \times d = d$.

Secondly, $b \times p = bp$; therefore $\frac{1}{b}$ of $\frac{d}{p}$ is (by *Sect.* 4. Chap. II.) $= \frac{d}{bp}$; Consequently $\frac{2d-q}{bp}$ is the Sum required.

Examples in Subſtraction.

1	$\frac{a+b+aa}{d}$	$\frac{a+b}{d+c}$	$\frac{2a}{d+a}$
2	$\frac{a+b}{d}$	$\frac{2b-a}{d+c}$	$\frac{a+b-d}{d+a}$
1 − 2 | 3	$\frac{aa}{d}$	$\frac{2a-b}{d+c}$	$\frac{a-b+d}{d+a}$

Let it be required to take $a + \frac{d}{p}$ from $\frac{a+c}{b+d}$.

First, $a + \frac{d}{p}$ is (by *Sect.* 2. Chap. II.) $= \frac{ap+d}{p}$.

Secondly, $\frac{a+c}{b+d}$ and $\frac{ap+d}{p}$ are (by *Sect.* 1. Chap. II.) equal to $\frac{pa+pc}{pb+pd}$ and $\frac{bap+bd+dap+d^2}{pb+pd}$ reſpectively.

Conſequently $\frac{pa+pc-bap-bd-dap-d^2}{pb+pd}$ is the Remainder required.

Again, Let it be required to take $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{b}{a}$ from $\frac{b}{a}$.

$\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{b}{a}$ is (by *Sect.* 4. Chap. II) $= \frac{3b}{8a}$.

And $\frac{b}{a}$ is $= \frac{8b}{8a}$

Therefore $\frac{8b-3b}{8a} = \frac{5b}{8a}$ is the Remainder ſought.

Note, *The univerſal manner of Adding and Subſtracting either Whole, or Fracted Quantities, is by + and — reſpectively.*

CHAP. V.

Multiplication of Fractional Quantities.

First prepare mix'd Quantities (if there be any to be Multiplyed) by reducing them to Fractions of the ſame Denomination. (by *Sect.* 1. Chap. II.) And whole Quantities by ſubſcribing an Unit under each of them : then

Rule.

Multiply the Numerators together for a new Numerator, and the Denominators together for a new Denominator ; then this new Fraction is the Product required.

Examples.

	1	$\frac{ab}{c}$	$\frac{3a-2b}{2d+c}$
	2	$\frac{d}{f}$	$\frac{4a+2b}{d}$
1 × 2	3	$\frac{abd}{cf}$	$\frac{12aa-2ab-4bb}{2dd+dc}$

Suppoſe it was required to Multiply $2a + \frac{b}{c} - 25$ by $3b + 4c$. Theſe prepar'd for the Work as above directed, will ſtand thus,

	1	$\frac{2ac+b-25c}{c}$
	2	$\frac{3b+4c}{1}$
1 × 2	3	$6bac + 3bb - \frac{75bc + 8acc}{c} + 4bc - 100cc$

N.B. *Any Fraction is Multiplyed by its Denominator, by caſting off, or taking away the Denominator.*

Thus $\frac{b}{a} \times a = b$; for $\frac{b}{a} \times \frac{a}{1} = \frac{ba}{a} = b$.

Again, Let it be required to Multiply $\frac{b}{c}$ of $\frac{a+b}{b+c}$ by f.

Firſt, $b \times \overline{a+b} \times f = baf + bbf$, the *Numerator* } of, or the
Sec. $c \times \overline{b+c} \times 1 = cb + cc$, the *Denomin.* }
Product required.

CHAP. VI.

Diviſion of Fractional Quantities.

THe Fractional Quantities being prepar'd as directed in the laſt Chapter, then

Rule.

Multiply the Numerator of the Dividend, by the Denominator of the Diviſor, for a new Numerator; and Multiply the Denominator of the Dividend by the Numerator of the Diviſor, for a new Denominator; ſo this new Fraction, is the Quotient required.

Examples.

Let $\frac{abd}{cf}$ be Divided by $\frac{ab}{c}$ the Work may ſtand

Thus $\frac{ab}{c}\Big)\,\frac{abd}{cf}\,\Big(\frac{abdc}{abcf} = \frac{d}{f}$ (by *Sect.* 3. Chap. II.)

Or thus $\frac{abd}{cf} \div \frac{ab}{c} = \frac{abdc}{abfc} = \frac{d}{f}$.

Or thus	1	$\frac{abd}{cf}$
	2	$\frac{ab}{c}$
1 ÷ 2	3	$\frac{abcd}{abcf} = \frac{d}{f}$

1 ÷ 2

	1	$\frac{a+b}{d}$	$\frac{aaa-bbb}{a+b}$
	2	$\frac{c-b}{a}$	$\frac{aa-ba+bb}{c}$
1÷2	3	$\frac{aa+ab}{dc-db}$	$\frac{aaac-bbbc}{aaa+bbb}$

Suppoſe it were required to divide $aa+\frac{3bba}{a+4b}$ by $a+b$.

The Work (when prepar'd as before directed) will ſtand thus

$$\frac{a+b}{1}\Big)\quad \frac{aaa+4aab+3abb}{a+4b}\quad \Big(\frac{aaa+4aab+3abb}{aa+5ba+4bb}$$

And $\frac{aaa+4aab+3abb}{aa+5ba+4bb}$ is $=\frac{aa+3ab}{a+4b}$ by *Sect.* 3. Chap. II.

When Fractions are of one Denomination, Caſt off the Denominators, and divide one Numerator by the other.

Thus, If $\frac{abbb}{c}$ were to be divided by $\frac{bb}{c}$ it will be $bb)\,abbb\,(ab$ the Quotient required.

$$\text{For } \frac{bb}{c}\Big)\quad \frac{abbb}{c}\quad \Big(\frac{abbbc}{bbc}=ab.$$

Again ſuppoſe it was required to Divide $\frac{aaa-abb}{c-d}$ By $\frac{aa+2ab+bb}{c-d}$.

By Caſting off $c-d$ from both, it will ſtand thus

$$aa+2ab+bb)\quad aaa-abb\quad \Big(\frac{aaa-abb}{aa+2ab+bb}=\frac{aa-ab}{a+b}$$

(by *Sect.* 3. Chap. II.)

PART III.

Involution.

CHAP. I.

Involution of whole Quantities.

Involution is the Raiſing or Producing of Powers, from any propos'd Root; and is perform'd in all reſpects like Multiplication; ſave only in this; Multiplication admits of any different Factors, but Involution ſtill Retains the ſame.

Examples.

	1	a	$-a$	*the Root, or ſingle Power.*
1 ⊙ 2	2	aa	$+aa$	*Square, or ſecond Power.*
1 ⊙ 3	3	aaa	$-aaa$	*Cube, or third Power.*
1 ⊙ 4	4	$aaaa$	$+aaaa$	*Biquadrat, or 4th Power.*
1 ⊙ 5	5	$aaaaa$	$-aaaaa$	*Surſolid, or 5th Power, &c.*

Note, The Figures plac'd in the Margent, after the Sign (⊙) of Involution; ſhew to what height the Root is Involved; and are called Indices of the Power; and are uſually placed over the Involved Quantities, in order to contract the Work, eſpecially when the Powers are any thing high.

$$\text{Thus}\begin{cases} a=a \\ a^2=aa \\ a^3=aaa \\ a^4=aaaa \end{cases} \qquad \text{And}\begin{cases} a^5=aaaaa \\ a^6=aaaaaa \\ a^5b^6=aaaaabbbbbb \\ a^2d^3b^4aadddbbbb \end{cases}$$

If the Quantities have Coefficients, the Coefficients muſt be Involved along with the Quantities. As in theſe.

Thus

Thus	1	$2a$	$-3a$	$5bc$
1 ⊖ 2	2	$4aa$	$+9aa$	$25bbcc$
1 ⊖ 3	3	$8aaa$	$-27aaa$	$125bbbccc$
1 ⊖ 4	4	$16aaaa$	$+81aaaa$	$625bbbbcccc$
1 ⊖ 5	5	$32aaaaaa$	$-243aaaaa$	$3125b^5c^5$. &c.

Involution of Compound Quantities is performed in the ſame manner, due regard being had to their Signs and Coefficients.

As for Inſtance, Suppoſe $a+b$ were given to be Multiplyed to the 5th Power.

Thus	1	$a+b$	called a *Binomial Root.*
		$a+b$	
1 × a	2	$aa+ab$	
1 × b	3	$+ab+bb$	
2 + 3, or 1 ⊖ 2	4	$aa+2ab+bb$	the *Square* of $a+b$
		$a+b$	
4 × a	5	$aaa+2aab+abb$	
4 × b	6	$aab+2abb+bbb$	
5 + 6, or 1 ⊖ 3	7	$aaa+3aab+3abb+bbb$	the *Cube* of $a+b$,
		$a+b$	
7 × a	8	$a^4+3a^3b+3a^2b^2+ab^3$	
7 × b	9	$a^3b+3a^2b^2+3ab^3+b^4$	
8 + 9, or 1 ⊖ 4	10	$a^4+4a^3b+6a^2b^2+4ab^3+b^4$	the *Biquadrat* (of $a+b$
		$a+b$	
10 × a	11	$a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4$	
10 × b	12	$a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5$	
[illegible] 4		$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$	the *Surſolid* or 5th Powr of $a+b$ required. &c.

Again, Let $a-b$ a *Reſidual Root*, be given to be Involved.

Then	1	$a-b$	
		$a-b$	
1 × a	2	$aa-ab$	
1 × −b	3	$-ab+bb$	
1 ⊖ 2	4	$aa-2ab+bb$	the *Square* of $a-b$
		$a-b$	

$4 \times a$	5	$a^3 - 2a^2b + abb$	
$4 \times - b$	6	$- a^2b + 2ab^2 - b^3$	
1 ⊙ 3	7	$a^3 - 3a^2b + 3ab^2 - b^3$	the *Cube* of $a - b$
		$a - b$	
$7 \times a$	8	$a^4 - 3a^3b + 3a^2b^2 - ab^3$	
$7 \times - b$	9	$- a^3b + 3a^2b^2 - 3ab^3 + b^4$	
1 ⊙ 4	10	$a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$	the 4*th Power* of $(a - b$
		$a - b$	
$10 \times a$	11	$a^5 - 4a^4b + 6a^3b^2 - 4a^2b^3 + ab^4$	
$10 \times - b$	12	$- a^4b + 4a^3b^2 - 6a^2b^3 + 4ab^4 - b^5$	
1 ⊙ 5	13	$a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$	the 5*th Power* of $a - b$, &c.

By comparing theſe two Examples together, you may make the following Obſervations.

1. That the Powers raiſ'd from a Reſidual Root, (*viz.* the Difference of the two Quantities) are the ſame with their like Powers raiſ'd from a Binomial Root, (or the Sum of two Quantities) ſave only in their Signs, *viz.* the Binomial Powers have the Sign + to every Term; but the Reſidual Powers have the Signs + and — interchangeably to every other Term.

2. The Indices of the Powers of the Leading Quantity (a) continually decreaſe in Arithmetical Progreſſion; *viz.* in the Square it is a^2, a^1; in the Cube a^3, a^2, a^1; in the Biquadrat a^4, a^3, a^2, a^1; &c.

3. The Indices of the other Quantity b, do continually increaſe in Arithmetical Progreſſion, *viz.* in the Square it is b^1, b^2; in the Cube b, b^2, b^3; in the Biquadrat b, b^2, b^3, b^4; &c.

4. The firſt and laſt Terms, are always pure Powers of the ſingle Quantities, and are both of the ſame height.

5. The Sum of the Indices of any two Letters join'd together, in the intermediate Terms are always equal to the Index of the higheſt Power, *viz.* of the firſt or laſt Term.

Theſe Obſervations being duly conſider'd, it will be eaſy to conceive, how the Terms of any propos'd Power, rais'd from a Binomial or Reſidual Root, muſt ſtand without their *Unciæ*, or Numeral Figures, or Coefficients.

For Inſtance, ſuppoſe it were required to raiſe the Binomial Root $a + b$, to the 7th Power; then the Terms of that Power will ſtand without their *Unciæ* in this Order,

Viz. $a^7 + a^6b + a^5b^2 + a^4b^3 + a^3b^4 + a^2b^5 + ab^6 + b^7$

And becauſe the *Unciæ* (not only of any ſingle Letter, but alſo) of every ſingle Power, how high ſoever it be, is an Unit or 1 (which neither Multiplies nor Divides) and all the Powers of any Binomial or Reſidual Root are naturally raiſed by Multiplying of the Precedent Power into its original Root, which is done by only joining each Letter in the Root to the Precedent Power with its *Unciæ*, and then removing the ſaid Power, when it is ſo join'd to the ſecond Letter, one place forwards (either to the Left, or Right Hand) it muſt needs follow,

That the *Unciæ* of the Second Term (in any ſuch Power) will always be the Sum of ſo many Units Added together more one, as there hath been Multiplications of the firſt Root; which will always be determined by the Index of the firſt Term in the Power.

And becauſe the *Unciæ* of all the intermediate Terms are only removed along with their Letters, it alſo follows; that if they are Added together, their reſpective Sums will produce the true *Unciæ* of the intermediate Terms in the new raiſed Power. As doth plainly appear from the following Numbers ſo removed without their Letters; which both ſhews and Demonſtrates an eaſie way of producing the *Unciæ* of any ordinary Power (*viz.* of one not very high) raiſed from either a Binomial, or Reſidual Root.

Thus

Add { 1 . 1 . The *Unciæ* of the firſt Root.
1 . 1

Add { 1 . 2 . 1 The *Unciæ* of the Square.
1 . 2 . 1

Add { 1 . 3 . 3 . 1 The *Unciæ* of the Cube.
1 . 3 . 3 . 1

Add { 1 . 4 . 6 . 4 . 1 The *Unciæ* of the 4th Power.
1 . 4 . 6 . 4 . 1

Add { 1 . 5 . 10 . 10 . 5 . 1 *Unciæ* of the 5th Power.
1 . 5 . 10 . 10 . 5 . 1

Add { 1 . 6 . 15 . 20 . 15 . 6 . 1 *Unciæ* of the 6th Power.
1 . 6 . 15 . 20 . 15 . 6 . 1

1 . 7 . 21 . 35 . 35 . 21 . 7 . 1 *Unciæ* of the 7th Power.

And ſo on in this manner *ad infinitum.*

Now if theſe Numbers be prefixt to the aforeſaid Letters, all the Terms will be compleated with their reſpective *Unciæ*, and will ſtand thus;

$$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

But that the buſineſs of finding theſe *Unciæ*, may be render'd yet more eaſy for Practice, it will be convenient to conſider what Series or Progreſſion the *Unciæ* of each Term do make from the aforeſaid Additions. Thus

Unciæ of the Firſt Term.	*Unciæ of the Second Term.*	*Unciæ of the Third Term.*	*Unciæ of the Fourth Term.*	*Unciæ of the Fifth Term.*	*Unciæ of the Sixth Term.*	*Unciæ of the Seventh Term.*	*Unciæ of the Eighth Term.* &c.	
1	1							*Unciæ of the ſingle Quantities.*
1	2	1						*Unciæ of the Square.*
1	3	3	1					*Unciæ of the Cube.*
1	4	6	4	1				*Unciæ of the 4th Power.*
1	5	10	10	5	1			*Unciæ of the 5th Power.*
1	6	15	20	15	6	1		*Unciæ of the 6th Power.*
1	7	21	35	35	21	7	1	*Unciæ of the 7th Power.* &c.

The *Unciæ* of the firſt Terms, is only a Series of Units, whoſe Sum is every where the *Unciæ* of the ſecond Term.

The *Unciæ* of the ſecond Term, is a Series of Numbers in Arithmetick Progreſſion; whoſe Sum is every where the *Unciæ* of the next ſuperiour Power in the third Term, and may be found by Theorem 3. Chap. I. Part VIII.

That is in the ſeventh Power it will be $\frac{\overline{6+1}\times 6}{2} = 21$ the *Unciæ* of the third Term.

The reſt of the *Unciæ* are a Compounded Series, whoſe reſpective Sums may be obtained from the *Unciæ* of their precedent Terms.

Thus $\frac{21 \times 5}{3} = 35$, then $\frac{35 \times 4}{4} = 35$.

Again $\frac{35 \times 3}{5} = 21$; alſo $\frac{21 \times 2}{6} = 7$, and $\frac{7 \times 1}{7} = 1$.

From hence may be deduc'd this general *Rule*.

Rule.

If the Index of the firſt Letter of any Term be Multiplyed into its own *Uncia*, and that Product be divided by the Number of Terms to that place, the Quotient will be the *Uncia* of the next ſucceeding Term forward.

That is, by the help of thoſe Indices that belong to the ſeveral Powers of the firſt or leading Letter only, (as a) the true *Unciæ* of every Term may be eaſily found.

Examples.

Let it be required to compleat all the Terms of the aforeſaid ſeveral Powers, *viz.* $a^7 + a^6b + a^5b^2 + a^4b^3 + a^3b^4 + a^2b^5 + ab^6 + b^7$ with their proper *Unciæ.*

1. The Index of a^7 (the firſt Term) to wit 7 will be the *Uncia* of the ſecond Term. Thus $a^7 + 7a^6b$.

2. Then half the ſecond Terms Index into its *Unciæ* *Viz.* $\frac{7 \times 6}{2} = 21$, will be the third Terms *Unciæ.*

Thus $a^7 + 7a^6b + 21a^5b^2$ will be the three firſt Terms.

3. Again $\frac{21 \times 5}{3} = 35$ is the *Uncia* of the fourth Term.

Then it will be $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3$.

4. Alſo $\frac{35 \times 4}{4} = 35$ will be *Unciæ* of the fifth Term.

Then it will be $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4$.

5. Again $\frac{35 \times 3}{5} = 21$ is the *Uncia* of the ſixth Term.

6. Alſo $\frac{21 \times 2}{6} = 7$ is the *Unciæ* of the ſeventh Term.

7. And $\frac{7 \times 1}{7} = 1$ is the *Unciæ* of the eighth Term.

Wherefore all the Terms, when compleated, ſtand Thus, $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$.

In like manner, if it be required to compleat all the Terms of $a^6 + a^5b + a^4b^2 + a^3b^3 + a^2b^4 + ab^5 + b^6$ (which are the Terms of the ſixth Power of $a + b$, without their *Unciæ*) it may be done thus,

1ſt. 1 is the *Uncia* of a^6; and 6 of a^5b.

2dly. $\frac{6 \times 5}{2} = 15$ is the *Unciæ* of the third Term.

Third,

3dly. $\frac{15 \times 4}{3} = 20$ is the *Unciæ* of the fourth Term.

4thly. $\frac{20 \times 3}{4} = 15$ is the *Unciæ* of the fifth Term.

5thly. $\frac{15 \times 2}{5} = 6$ is the *Unciæ* of the ſixth Term.

6thly. $\frac{6 \times 1}{6} = 1$ is the *Unciæ* of the ſeventh Term.

Therefore all the Terms when Compleated, ſtand thus,
$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.

Now here it may be further obſerved, that the *Unciæ* do only increaſe until the Indices of the two Letters become equal, or change Places; and the reſt of the *Unciæ* will return, or decreaſe in the ſame Order. That is, wherever the Indices of the Letters are alike, there the *Unciæ* will be alike.

And therefore one needs to find the *Unciæ* (as before) but to half the Numbers of Terms in any Power.

Corollary.

If m be equal to any whole Number; then the ſeveral Terms of the m^{th} Power of $a + b$ will be $= a^m + ma^{m-1}b + m \times \frac{m-1}{2}a^{m-2}b^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3}a^{m-3}b^3 +$ &c.

Any 3, 4, 5, &c. Simple Quantities connected by the Signs +, −, may be involved in like manner with the * foregoing. But here ought rather to be Involved to any required Power, by the ſame Power of a Binomial, in order to know the Nature and Compoſition of Powers, and thence the manner of Evolving them. Thus

Suppoſe it were required to Involve $g + h - i$, to the third Power.

The third Power of the Binomial $a + b$ is $a^3 + 3aab + 3abb + b^3$, which is your Canon for Involving.

Then $g = a$ (i. e. 1ſt. a)
$h = b$ (i. e. 1ſt. b)

And therefore $g^3 = a^3$
$3ggh = 3aab$
$3ghh = 3abb$
$h^3 = b^3$

Or thus: put $g+h=a$, and ſo $\overline{g+h-i} = a-i$, and $\overline{g+h-i}^3 = a^3 - 3a^2i + 3ai^2 - i^3$; here reſtore $g+h$ for a, and there will be

$$\left.\begin{array}{r} g^3+3g^2h+3gh^2+h^3 = a^3 \text{---} \\ -3g^2i-6ghi-3h^2i = -3a^2i \\ +3gi^2+3hi^2 = +3ai^2 \end{array}\right\} = 3^d \text{ Power of } g+h-i$$

g^3

---

* That is by Multiplication.

$$\begin{array}{rrrrl} g^3 + 3ggh & + 3ghh & + h^3 & = a^3 \\ - 3ggi & - 6ghi & - 3hhi & = 3aab \\ & + 3gii & + 3hii & = 3abb \\ & & - iii & = b^3 \end{array}$$

Here a (i. e. 2*d.* a) $= g + h$. And 2*d.* $b = - i$.

$$\begin{array}{llll} g^3 \begin{array}{l} + 3h \\ - 3i \end{array} gg & \begin{array}{l} + 3hh \\ - 6hi \\ + 3ii \end{array} g & \begin{array}{l} + h^3 \\ - 3hhi \\ + 3hii \\ - i^3. \end{array} \end{array}$$

Is the Cube of $g + h - i$, which was required.

CHAP. II.

Involution of Numbers.

ANy abſolute Number being firſt reduc'd into its ſeveral Members (which are the ſeveral Significant Figures in the given abſolute Number with their due Number of Cyphers after each of them, and before ſuch of them as are Decimals, along with the Point or Decimal Character) may be Involved to any required Power, by the help of the ſame Power of a Binomial; (as in the foregoing *Example* in Specie, which ſhews and demonſtrates the manner of doing it) always obſerving to begin with the greateſt Members of the ſaid Number.

Examples.

1. Let it be required to find the Square of 5709.

1. The Square of the Binomial $a + b$, is $aa + 2ab + bb$, which is your Canon for Involving.

2. $5709 = 5000 + 700 + 9$

Operation.

$$\begin{array}{rrcl} & 5000 & = & a \quad \text{(i. e. 1ſt. } a\text{)} \\ & 700 & = & b \quad \text{(i. e. 1ſt. } b\text{)} \\ \hline (5000 \times 5000 =) & 25000000 & = & aa \\ (2 \times 5000 \times 700 =) & 7000000 & = & 2ab \\ (700 \times 700 =) & 490000 & = & bb \\ \hline (5700 \times 5700 =) & 32490000 & = & aa \\ (2 \times 5700 \times 9 =) & 102600 & = & 2ab \\ (9 \times 9 =) & 81 & = & bb \\ \hline & 32592681 & & \end{array}$$

Here a (i. e. 2*d.* a) $= 5700$. And b (i. e. 2*d.* b) $= 9$.

32592681 the Square required.

2. What is the Cube of 463?

The Cube of the Binomial $a + b$, is $aaa + 3aab + 3abb + bbb$ Canon.

$463 = 400 + 60 + 3.$

Operation.

$400 = a$
$60 = b$

$64000000 = aaa$
$28800000 = 3a^2b$
$4320000 = 3ab^2$
$216000 = b^3$

$97336000 = a^3$
$1904400 = 3a^2b$
$12420 = 3ab^2$
$27 = b^3$

{ Here $a = 460$.
{ And $b = 3$.

99252847 the Cube of 463 Anſwer.

Scholium.

The Square Cube, &c. of any Number with Cyphers after it to the place of Units incluſive, will have twice thrice, &c. reſpectively the ſaid Number of Cyphers after it; wherefore in Involving, or Evolving Numbers; the following Table will be aſſiſting.

The Root.	1	2	3	4	5	6	7	8	9
Square.	1	4	9	16	25	36	49	64	81
Cube	1	8	27	64	125	216	343	512	729
Biquadrat.	1	16	81	256	625	1296	2401	4096	6561
5th Power.	1	32	243	1024	3125	7776	16807	32768	59049

&c.

3. What is the Square of 1 . 21?

$1.21 \text{ is} = 1 + .2 + .01$

$1 = aa$
$.4 = 2ab$
$.04 = bb$

{ Here $a = 1$,
{ and $b = .2$.

$1.44 = aa$
$.024 = 2ab$
$.0001 = bb$

{ Here $a = 1.2$
{ and $b = .01$.

1.4641 Anſwer.

5. Take one Example of a mixt Involution of Numbers, the Converſe of which is called the Numeral Exegeſis.

If $y = 123$, what is the Value of $y^3 + y$?

Suppoſe $a + b = y$.

Then $a^3 + 3a^2b + 3ab^2 + b^3 = y^3$.

And $a + b = y$.

Therefore $\left.\begin{array}{l} a^3 + 3a^2b + 3ab^2 + b^3 \\ + a + b \end{array}\right\}$ Is the Canon for Involving.

And $123 = 100 + 20 + 3$.

Operation.

$$\begin{array}{rcl}
100 & = & a \\
20 & = & b \\ \hline
1000000 & = & a^3 \\
100 & = & a \\
600000 & = & 3a^2b \\
20 & = & b \\
120000 & = & 3ab^2 \\
8000 & = & b^3 \\ \hline
1728120 & = & a^3 + a \\
129600 & = & 3aab \\
3 & = & b \\
3240 & = & 3ab^2 \\
27 & = & b^3 \\ \hline
1860990 & = & y^3 + y
\end{array}$$

Here $a = 120$, And $b = 3$.

Anſwer.

Note, *that moſt Men do not make uſe of the foregoing Cyphers; but, if they are not Written, they muſt be underſtood as Written.*

CHAP. III.

Involution of Fractions.

THe *Rule* for Involving Fractional Quantities or Numbers, is this; *viz.*

Involve the Numerator into it ſelf for a new Numerator, and the Denominator into it ſelf for a new Denominator; each ſo often as the Power requires.

Thus	1	$\frac{b}{a}$	$\frac{3bc}{2ad}$	$\frac{b + d}{a - c}$
1@2	2	$\frac{bb}{aa}$	$\frac{9b^2c^2}{4a^2d^2}$	$\frac{b^2 + 2bd + dd}{aa - 2ac + cc}$
1@3	3	$\frac{b^3}{a^3}$	$\frac{27b^3c^3}{8a^3d^3}$	$\frac{b^3 + 3b^2d + 3bd^2 + d^3}{a^3 - 3a^2c + 3ac^2 - c^3}$

If x be equal to $\frac{g + h}{l - k}$ then $x^3 - gx$ will be equal to $\frac{g^3 + 3g^2h + 3gh^2 + h^3 - gg - gh}{l^3 - 3l^2k + 3lk^2 - k^3 - gl + gk}$.

again	1	$\frac{1}{463}$	$\frac{68}{5709}$
1@2	2	$\frac{1}{214369}$	$\frac{4624}{32592681}$
1@3	3	$\frac{1}{99252847}$	$\frac{314432}{186071615829}$

If x be equal to $\frac{132}{1000}$

Then $2x^2 + 3x$ will be equal to $\frac{35244}{2003000}$.

PART IV.

Evolution.

CHAP. I.

Evolution of whole Quantities.

EVolution is the Extracting of Roots from any given Power; that is, it is the converſe Work to that of Involution; and in ſingle Quantities it is eaſy, if the given Power have ſuch a Root as is required, which may be thus known,

If the given Power have no Numbers prefixt to it, and its Index can be Divided by the Number Denominating the Root required, the Quotient will be the Index of the Root ſought.

Thus, If the Cube Root of *aaaaaa*, *viz.* a^6 were required, (the Number that Denominates the Cube-Root is 3) then 3) 6 (2; that is $\sqrt[3]{a^6} = a^2$ the Root required. And ſuch Operations are uſually ſet down.

Thus	1	a^6	a^6b^6	$a^6b^6c^6$
1 ɯ 2	2	a^3	a^3b^3	$a^3b^3c^3$
1 ɯ 3	3	a^2	a^2b^2	$a^2b^2c^2$
3 ɯ 2	4	a	$a\,b$	$a\,b\,c$

Note, *The Figures plac'd in the Margin after the Sign* (ɯ) *of Evolution, denote the Number Denominating the Root to be Extracted.*

If the given Powers have Coefficients (*viz.* Numbers prefixed to them,) then you muſt Extract their reſpective Roots.

Thus	1	$81\,a^4$	$1296\,a^8b^8$	$20736\,a^4b^4c^4$
1 ɯ 2	2	$9\,a^2$	$36\,a^4b^4$	$144\,a^2b^2c^2$
1 ɯ 4	3	$3\,a$	$6\,a^2b^2$	$12\,a\,b\,c$
2 ɯ 2	4	$3\,a$	$6\,aabb$	$12\,a\,b\,c$

But if the Root required cannot be truly Extracted out of both the Coefficients and Indices of the given Power, then it is a Surd, and muſt have the Sign of the Root required prefixt to it (or its Index over it.) *See* Part V.

Thus	1	a^5	$67aaa$	$216\,b^3d^3$
1 ∞ 2	2	$^2\sqrt{a^5}$	$^2\sqrt{67aaa}$	$^2\sqrt{216\,b^3d^3} = \sqrt{216\,b^3d^3}$
1 ∞ 3	3	$^3\sqrt{a^5}$	$^3\sqrt{67aaa}$	$6\,b\,d$

Evolution of Compound Quantities may be perform'd by the following

Rule.

Firſt Involve any Binomial, as $x + y$, to ſuch a Power as the Root to be Extracted requires; that is, to the ſecond Power, if it be the Square Root; to the third Power, if it be the Cube-Root; &c. that is to be Extracted. And the Power thus produc'd, is your Canon for Evolving, which is to be us'd in the following Manner, *viz.*

Firſt find the Root of the firſt or greateſt Term of the given Compound Quantity, and call it x, then Subſtract the ſaid Term from the ſaid Compound Quantity; the Remainder call your *Reſolvend*; and the Value of the Coefficient of y in the ſecond Term, of your Canon, call your *Diviſor*. By this Divide the *Reſolvend*, and the firſt Term of the Quotient is $= y$. *But here Note*, that in the continuance of the Operation, you may make more of the firſt Terms of the Quotient $= y$. Then find the Sum of the Values of all the Terms of your Canon but the firſt; and call it your *Ablativum*, which Subſtract from your *Reſolvend*; and call the Remainder, if there be any, your ſecond *Reſolvend*. The Sum of the Values of the next foregoing x and y thus found, call x, (*i. e.* your next x); and proceed with this Value of x, in order to find that of the next y as before is taught; and ſo on.

Example.

1. Suppoſe it was required to Extract the Square-Root of

$$a^4 \begin{array}{l} -6n \\ +2x \end{array} a^3 \begin{array}{l} +9n^2 \\ -6nx \\ +xx \end{array}$$

1*ſt*. $\overline{x+y} \times \overline{x+y} = xx + 2xy + yy$, is your Canon for Extracting the Square-Root.

2dly. The Square-Root of aa, (the firſt Term of the given Compound Quantity) is a; and therefore $a = x$ (*i. e.* 1*ſt.* x); and $a^2 = x^2$ which taken from the ſaid Compound Quantity, leaves $\begin{matrix}-6n\\+2x\end{matrix} a \begin{matrix}+9nn\\-6nx\\+xx\end{matrix}$ for a Reſolvend. The Coefficients of y in the ſecond Term of the Canon, is $2x$, and its Value $2a$ is your *Diviſor*; by which the Reſolvend being Divided, the firſt Term of the Quotient is $\begin{matrix}-3n\\+x\end{matrix}$; wherefore $2xy + yy = \begin{matrix}-6n\\+2x\end{matrix} a \begin{matrix}+9nn\\-6nx\\+xx\end{matrix}$ is the *Ablativum*, which taken from the ſaid Reſolvend, leaves 0: conſequently $a \begin{matrix}-3n\\+x\end{matrix}$ is the Root ſought.

Operation:

$$a^2 \begin{matrix}-6n\\+2x\end{matrix} a \begin{matrix}+9nn\\-6nx\\+xx\end{matrix} \quad \left(a \begin{matrix}-3n\\+x\end{matrix} \right. \text{Root.}$$

$$a^2 = xx$$

$$\begin{matrix}-6n\\+2x\end{matrix} a \begin{matrix}+9nn\\-6nx\\+xx\end{matrix} \quad \textit{Reſolvend.}$$

$$\textit{Diviſor}\ \ 2a = 2x$$

$$\textit{Ablativum}\left\{\begin{array}{l} \begin{matrix}-6n\\+2x\end{matrix} a = 2xy. \\ \\ \left.\begin{matrix}+9nn\\-6nx\\+xx\end{matrix}\right\} = yy. \end{array}\right. \qquad \text{Here } y = \begin{matrix}-3n\\+x\end{matrix}$$

$$0 \quad \textit{Remainder.}$$

Or the ſame Root may be Extracted in an eaſier, but more tedious manner; thus

$$a^2 - 6an + 9nn + 2ax - 6nx + xx \quad (a - 3n + xx \text{ Root.}$$
$$a^2 = xx$$

$$-6an + 9n^2 + 2ax - 6nx + xx \quad \textit{Reſolvend.}$$

$$\textit{Diviſor}\ 2a = 2x$$

$$\begin{array}{l} -6an = 2xy \\ +9nn = yy \end{array} \qquad \text{Here } y = -3n.$$

$$-6an + 9nn \quad \textit{Ablativum.}$$

$$0 \quad + 2ax - 6nx + xx \quad \textit{Reſolvend.}$$

$$\textit{Diviſor} \quad 2a - 6n = 2x$$

$$\begin{array}{r} 2ax - 6nx = 2xy \\ + xx = yy \end{array} \qquad \left\{ \begin{array}{l} \text{Here } x \text{ (i. e. 2d. } x\text{)} \\ = a - 3n. \text{ And } y \\ \text{(i. e. 2d. } y\text{)} = x. \end{array} \right.$$

$$2ax - 6nx + xx \quad \textit{Ablativum.}$$

$$0 \quad \textit{Remainder.}$$

But if the given Quantity hath not ſuch a Root as is required, then the Evolution may be continued to an endleſs Series. Thus

2. Let it be required to Extract the Square-Root of $rr \pm xx$, very near.

Operation.

$$rr \pm zz \left(r \pm \frac{zz}{2r} - \frac{z^4}{8r^3} \pm \frac{z^6}{16r^5} - \frac{5z^8}{128r^7}, \text{ \&c. } \textit{Sine} \right.$$

$$rr = xx \qquad \textit{(Fine.}$$

$$0 + zz \quad \textit{Reſolvend.}$$

$$\textit{Div.} \quad 2r = 2x$$

$$\textit{Ab.} \left\{ \begin{array}{l} \pm zz = 2xy \\ + \dfrac{z^4}{4r^2} = yy \end{array} \right. \qquad \text{Here } y = \frac{+z^2}{2r}$$

$$-\frac{z^4}{4r^2} \quad \textit{Reſolvend.}$$

$$\textit{Diviſor} \quad 2r \pm \frac{z^2}{r} = 2x. \quad \text{Here } x = r \pm \frac{z^2}{2r}$$

$$\textit{Ab.} \left\{ \begin{array}{l} -\dfrac{z^4}{4r^2} \mp \dfrac{z^6}{8r^4} = 2xy. \quad \text{Here } y = \dfrac{-z^4}{8r^3} \\ + \dfrac{z^8}{64r^6} = yy. \end{array} \right.$$

$\pm$

$$\pm\frac{z^6}{8r^4} - \frac{z^8}{64r^6}$$ *Reſolvend.*

Diviſor $2r \pm \frac{z^2}{r} - \frac{z^4}{4r^3} = 2x$. Here $x = r \pm \frac{z^2}{2r} - \frac{z^4}{8r^3}$

Ablat. $$\left\{\begin{array}{l} \pm\frac{z^6}{8r^4} + \frac{z^8}{16r^6} \mp \frac{z^{10}}{64r^8} = 4xy. \text{ Here } y = \pm\frac{z^6}{16r^5} \\ \qquad\qquad + \frac{z^{12}}{256r^{10}} = yy. \end{array}\right.$$

Reſolvend $-\frac{5z^8}{64r^6} \pm \frac{z^{10}}{64r^8} - \frac{z^{12}}{256r^{10}}$.

Diviſor $2r \pm \frac{z^2}{r} - \frac{z^4}{4r^3} \pm \frac{z^6}{8r^5} = 2x$.

Now if by this Diviſor, you Divide the next foregoing Reſolvend, you'll find $-\frac{5z^8}{128r^7} \pm \frac{7z^{10}}{256r^9} - \frac{21z^{12}}{1024r^{11}} \pm \frac{33z^{14}}{2048r^{13}}$ for the Value of the next y, which added to $r \pm \frac{z^2}{2r} - \frac{z^4}{8r^3} \pm \frac{z^6}{16r^5}$, the Value of the next foregoing x, is near = the Square-Root of $r^2 \pm z^2$.

N. B. *The Sign* ± *denotes* + *or* −, *and the Sign* ∓ *denotes* − *or* +.

Corollary.

Here you may ſee that the *Unciæ* of the Square-Root of a Binomial or Reſidual, are $= 1,\ \frac{1}{2},\ \frac{1}{2} \times \frac{\frac{1}{2} - 1}{2},\ \frac{1}{2} \times \frac{\frac{1}{2} - 1}{2} \times \frac{\frac{1}{2} - 2}{3}$, &c; that is, putting n = Index of the Square-Root or $= \frac{1}{2}$, the *Unciæ* of the Square-Root of a Binomial or Reſidual are $= 1,\ n,\ n \times \frac{n-1}{2},\ n \times \frac{n-1}{2} \times \frac{n-2}{3},\ n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c. *In infinitum.*

3. Let it be required to Extract the Cube-Root of $r^3 \pm z^3$, very near.

$\overline{x+y} \times \overline{x+y} \times \overline{x+y} = xxx + 3xxy + 3xyy + yyy$ Canon.

$-6an = 2xy$ Here $y = -3n$.

$+9nn = yy$

$-6an + 9nn$ *Ablativum.*

$o + 2ax - 6nx + xx$ *Reſolvend.*

Diviſor $2a - 6n = 2x$ $\left\{\begin{array}{l}\text{Here } x \ (i.e.\ 2d.\ x) \\ = a - 3n.\ \text{And } y \\ (i.e.\ 2d.\ y) = x.\end{array}\right.$

$2ax - 6nx = 2xy$

$+ xx = yy$

$2ax - 6nx + xx$ *Ablativum.*

o *Remainder.*

But if the given Quantity hath not ſuch a Root as is required, then the Evolution may be continued to an endleſs Series. Thus

2. Let it be required to Extract the Square-Root of $rr \pm xx$, very near.

Operation.

$$rr \pm zz \left(r \pm \frac{zz}{2r} - \frac{z^4}{8r^3} \pm \frac{z^6}{16r^5} - \frac{5z^8}{128r^7}, \right.$$ &c. *Sine (Fine.*

$rr = xx$

$o + zz$ *Reſolvend.*

Div. $2r = 2x$

Ab. $\left\{\begin{array}{l} \pm zz = 2xy \\ + \dfrac{z^4}{4r^2} = yy \end{array}\right.$ Here $y = \dfrac{+z^2}{2r}$

$-\dfrac{z^4}{4r^2}$ *Reſolvend.*

Diviſor $2r \pm \dfrac{z^2}{r} = 2x$. Here $x = r \pm \dfrac{z^2}{2r}$

Ab. $\left\{\begin{array}{l} -\dfrac{z^4}{4r^2} \mp \dfrac{z^6}{8r^4} = 2xy. \\ + \dfrac{z^8}{64r^6} = yy. \end{array}\right.$ Here $y = \dfrac{-z^4}{8r^3}$

$\pm$

$$\pm\frac{z^6}{8r^4} - \frac{z^8}{64r^6} \quad \textit{Reſolvend.}$$

$$\textit{Diviſor}\quad 2r \pm \frac{z^2}{r} - \frac{z^4}{4r^3} = 2x. \quad \text{Here } x = r \pm \frac{z^2}{2r} - \frac{z^4}{8r^3}$$

$$\textit{Ablat.}\begin{cases} \pm\dfrac{z^6}{8r^4} + \dfrac{z^8}{16r^6} \mp \dfrac{z^{10}}{64r^8} = 2xy. \quad \text{Here } y = \pm\dfrac{z^6}{16r^5} \\ \qquad\qquad + \dfrac{z^{12}}{256r^{10}} = yy. \end{cases}$$

$$\textit{Reſolvend}\quad -\frac{5z^8}{64r^6} \pm \frac{z^{10}}{64r^8} - \frac{z^{12}}{256r^{10}}.$$

$$\textit{Diviſor}\quad 2r \pm \frac{z^2}{r} - \frac{z^4}{4r^3} \pm \frac{z^6}{8r^5} = 2x.$$

Now if by this Diviſor, you Divide the next foregoing Reſolvend, you'll find $-\frac{5z^8}{128r^7} \pm \frac{7z^{10}}{256r^9} - \frac{21z^{12}}{1024r^{11}} \pm \frac{33z^{14}}{2048r^{13}}$, for the Value of the next y, which added to $r \pm \frac{z^2}{2r} - \frac{z^4}{8r^3} \pm \frac{z^6}{16r^5}$, the Value of the next foregoing x, is near $=$ the Square-Root of $r^2 \pm z^2$.

N. B. *The Sign* ± *denotes* + *or* —, *and the Sign* ∓ *denotes* — *or* +.

Corollary.

Here you may ſee that the *Unciæ* of the Square-Root of a Binomial or Reſidual, are $= 1,\ \frac{1}{2},\ \frac{1}{2}\times\frac{\frac{1}{2}-1}{2},\ \frac{1}{2}\times\frac{\frac{1}{2}-1}{2}\times\frac{\frac{1}{2}-2}{3}$, &c; that is, putting $n =$ Index of the Square-Root or $= \frac{1}{2}$, the *Unciæ* of the Square-Root of a Binomial or Reſidual are $= 1,\ n,\ n\times\frac{n-1}{2},\ n\times\frac{n-1}{2}\times\frac{n-2}{3},\ n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$, &c. *In infinitum.*

3. Let it be required to Extract the Cube-Root of $r^3 \pm z^3$, very near.

$$\overline{x+y}\times\overline{x+y}\times\overline{x+y} = xxx + 3xxy + 3xyy + yyy \quad \text{Canon.}$$

Operation.

$$\begin{array}{l} r^3 \pm z^3 \\ r^3 = x^3 \end{array} \left(r \pm \frac{z^3}{3r^2} - \frac{z^6}{9r^5}, \text{ \&c.} \right.$$

$$0 \pm z^3 \quad \textit{Resolvend.}$$

$$\textit{Divisor } 3r^2 = 3x^2$$

$$\textit{Ablativum} \begin{cases} \pm z^3 = 3x^2y \\ + \dfrac{z^6}{3r^3} = 3xy^2. \\ \pm \dfrac{z^9}{27r^6} = y^3. \end{cases} \quad \text{Here } y = \pm \frac{z^3}{3r^2}$$

$$-\frac{z^6}{3r^3} \mp \frac{z^9}{27r^6} \quad \textit{Resolvend.}$$

$$\textit{Divisor } 3r^2 \pm \frac{2z^3}{r} + \frac{z^6}{3r^4} = 3x^2.$$

$$\textit{Ab.} \begin{cases} -\dfrac{z^6}{3r^3} \mp \dfrac{2z^9}{9r^6} - \dfrac{z^{12}}{27r^9} = 3x^2y \\ + \dfrac{z^{12}}{27r^9} \pm \dfrac{z^{15}}{81r^{12}} = 3xyy \\ - \dfrac{z^{18}}{729r^{15}} = y^3 \end{cases} \left\{ \begin{array}{l} \text{Here } x = r \pm \dfrac{z^3}{3rr}, \\ \text{and } y = -\dfrac{z^6}{9r^5}, \text{ or} \\ \text{it may be} = -\dfrac{z^6}{9r^5} \\ \pm \dfrac{5z^9}{81r^8}. \end{array} \right.$$

$$\pm \frac{5z^9}{27r^6} \ * \mp \frac{z^{15}}{81r^{12}} + \frac{z^{18}}{729r^{15}} \quad \textit{Resolvend.}$$

$$\textit{Divisor } 3r^2 \pm \frac{2z^3}{r} - \frac{z^6}{3r^4} \mp \frac{2z^9}{9r^7} + \frac{z^{12}}{27r^{10}} = 3x^2.$$

In this laſt Diviſor x is $= r \pm \frac{z^3}{3rr} - \frac{z^6}{9r^5}$. By which Diviſor, if you Divide its reſpective Reſolvend, you'll have $\pm \frac{5z^9}{81r^8} - \frac{10z^{12}}{243r^{11}} \pm \frac{22z^{15}}{729r^{14}}$ for the Value of the next y.

And therefore the Cube Root of $r^3 \pm z^3$ is near $= r \pm \frac{z^3}{3r^2} - \frac{z^6}{9r^5} \pm \frac{5z^9}{81r^8} - \frac{10z^{12}}{243r^{11}} \pm \frac{22z^{15}}{729r^{14}}$.

Corollary.

Here you may ſee that the *Unciæ* of the Cube-Root of a Binomial or Reſidual are $1,\ \frac{1}{3},\ \frac{1}{3} \times \frac{\frac{1}{3} - 1}{2},\ \frac{1}{3} \times \frac{\frac{1}{3} - 1}{2} \times \frac{\frac{1}{3} - 2}{3}$, &c. *In infinitum*; that is, putting $n =$ Index of the Cube-Root $= \frac{1}{3}$.) the *Unciæ* of the Cube-Root of a Binomial or Reſidual, are $= 1,\ n,\ n \times \frac{n-1}{2},\ n \times \frac{n-1}{2} \times \frac{n-2}{3},\ n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c. *Sine Fine.*

Scholium.

In like manner you may find that putting $n = \frac{1}{4}$. the *Unciæ* of the Biquadrat-Root of a Binomial or Reſidual, are $= 1,\ n,\ n \times \frac{n-1}{2},\ n \times \frac{n-1}{2} \times \frac{n-2}{3},\ n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c. *In infinitum.* And ſo on for ſuperiour Roots; whence, and from what has been ſaid in Part III. Chap. I. we have good Reaſon to believe that Univerſally the *Unciæ* of any Binomial or Reſidual, whoſe Index is $= n$, are $= 1,\ n,\ n \times \frac{n-1}{2},\ n \times \frac{n-1}{2} \times \frac{n-2}{3},\ \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c. But for a further Confirmation of this, I refer you to Part XV. Chap. I.

CHAP. II. and III.

Evolution and mixt Evolution of Numbers.

NOte, *What I call mixt Evolution, is the Method of Extracting the Roots of adfected Æquations.*

Rule.

When you are to Extract any unmixt Root; *viz.* the Square-Root, Cube-Root; &c. of a given Number; Involve any Binomial $x + y$ into it ſelf according to the Number Denominating the Root to be Extracted; and the Power thus produc'd is your Canon for Evolving. But when you are to Extract any of the Roots of an adfected Æquation, ſuppoſe the Binomial $x + y =$ the Root you ſeek; then inſtead of the ſaid Root (or unknown Letter) and its Powers in the Æquation, ſubſtitute their reſpective Values, *viz.* $x + y$ and the reſpective Powers thereof. And the Sum of the Terms wherein either x or y occurs, in the Æquation thus had, is your Canon for Evolving the abſolute Number in this (or in the propos'd) Æquation.

Having thus fram'd a Canon for Evolving, the Operation is to be perform'd in the following Manner, *viz.*

1*ſt.* Find the firſt or greateſt Member (*viz.* the firſt Significant Figure, with its due number of Cyphers) of the Root ſought; and call it x; then having found the Value of the firſt Term, or of the Sum of the firſt Terms of your Canon, *i.e.* of all thoſe Terms wherein x and its Powers only occur, Subſtract it from the abſolute Number; the Remainder call your *Reſolvend.* And the Value of the Coefficient, or of the Sum of the Coefficients of y in the ſecond Term or Terms of your Canon, call your *Diviſor.* Now by Dividing the ſaid *Reſolvend* by this *Diviſor*, the Value of y, or the 2*d.* Member of the ſaid Root is in ſome Caſes found, but not in all; wherefore in the beginning of your Operation, you muſt take care that y be $=$ the greateſt Member, and that the Sum of the Values of all the Terms of your Canon wherein y occurs, may not exceed the ſaid Reſolvend. Having thus found the Value of y, as alſo the Sum of the Values of all the Terms of your Canon, wherein y occurs; place the former along with the before found Value of x in the Root, and call the latter your *Ablativum*; which Subſtract from the ſaid *Reſolvend*; and, if there be a Remainder, call it your next *Reſolvend.* The Sum of the Values of the next foregoing x and y, call x, (*i.e.* a 2*d.* x nearer the truth). And proceed with this Value of x, in order to find that of the next y ~~as before is taught. And thus proceed till the *Ablativum* taken~~ from its Reſolvend leaves 0; or 'till you have as many Decimal Figures as you think ſufficient.

Note, *tho' by the firſt Diviſion, you may not find the next Member of the Root ſought; yet in continuing the Operation, one Diviſion may ſerve to find ſeveral of the next following Members, or the Value of y to many places of Figures, as will appear in the laſt Part of this Part IV.*

1. Suppoſe it was required to Extract the Square Root of 1024.

1ſt. $\overline{x+y} \times \overline{x+y} = xx + 2xy + yy$, is your Canon for Extracting the Square-Root.

2dly. The * greateſt Member of the Square-Root of 4624 is 60 (for $70 \times 70 = 4900$); therefore $x = 60$, and $xx = 3600$, which taken from 4624, leaves 1024 for a Reſolvend, and the Coefficient of y in the ſecond Term of your Canon is $2x = 120$ for your Diviſor; by which Dividing the Reſolvend, viz. 1024, the Quotient is $8 = y$; therefore $2xy + yy = 1024$ is the *Ablativum*, which taken from the Reſolvend, leaves 0, Whence the Square-Root of 4624 is $60 + 8 = 68$.

See the Table in Page 42.

Operation,

4624 ($60 + 8 = 68$ the Root required.
$3600 = xx$
1024 *Reſolvend.*
Div. $120 = 2x$
$960 = 2xy$
$64 = yy$ { Here $y = 8$.
1024 *Ablativum.*
0

2. Let it be required to Extract the Cube Root of 99252847.
$\overline{x+y} \times \overline{x+y} \times \overline{x+y} = xxx + 3xxy + 3xyy + y^3$ Canon.

Operation.

99252847 ($400 + 60 + 3 = 463$ the Root required.
$64000000 = x^3$ { Here $x = 400$.
Reſol. 35252847
* *Divi.* $480000 = 3x^2$
$28800000 = 3x^2y$ Here $y = 60$
$4320000 = 3xy^2$
$216000 = y^3$
33336000 *Ablativum.*
1916847 *Reſolvend.*
Diviſor $634800 = 3x^2$ Here $x = 460$.

See the Table in Page 42.

* *Here you may ſee that by Dividing the Reſolvend by this Diviſor, the Value of y thereby found, would be = 70, but the Ablativum ſo produc'd, would exceed the Reſolvend, wherefore y = 60.*

1904400

$1904400 = 3x^2y$ { Here $y = 3$.
$12420 = 3xyy$
$27 = yyy$

1916847 *Ablativum.*

0 *Remainder.*

9. Suppose it was required to Extract the Biquadrat Root of 66121111747853987761.

$\overline{x+y} \times \overline{x+y} \times \overline{x+y} \times \overline{x+y} = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ Canon.

Operation.

66121111747853987761 (50000 + 700 + 9 =
$62500000000000000000 = x^4$ (50709 the Root required.

3621111747853987761 *Resolvend.* Here $x = 50000$.

Div. $500000000000000 = 4x^3$

$3500000000000000000 = 4x^3y$. Here $y = 700$.
$73500000000000000 = 6x^2yy$
$686000000000000 = 4xy^3$
$240100000000 = yyyy$

3574188401000000000 *Ablativum.*

4692907753987761 *Resolvend.*

$521295372000000 = 4x^3$. Here $x = 50700$.

$4691658348000000 = 4x^3y$. Here $y = 9$.
$12492581400000 = 6x^2y^2$
$147841200 = 4xy^3$
$6561 = y^4$

4692907753987761 *Ablativum.*

0 *Remainder.*

If the given Power hath not ſuch a Root as is required, you may notwithſtanding find a Root nearer the truth, than any aſſigned in the following manner.

Suppoſe it was required to find the Square Root of 2 very near.

Operation.

2 (1 + .4 + .01 + .004 + .0002 + &c. = 1.4142 +

1

$1 = xx$

Diviſor $2 = 2x)$ 1 *Reſolvend.*

$.8 = 2xy$. Here $y = .4$.
$.16 = yy$

.96 *Ablativum.*

Div. $2.8 = 2x)$ .04 *Reſolvend.*

$.028 = 2xy$ {Here $x = 1.4$.
$.0001 = yy$ {And $y = .01$.

.0281 *Ablativum.*

Div. $2.82 = 2x)$ .0119 *Reſolvend.*

$.01128 = 2xy$ {Here $x = 1.41$.
$.000016 = yy$ {And $y = .004$.

.011296 *Ablativum.*

Div. $2.828 = 2x)$ .000604 *Reſolvend.*

$.0005656 = 2xy$ {Here $x = 1.414$.
$.00000004 = yy$ {And $y = .0002$.

.00056564 *Ablativum.*

.00003836 *Remainder.*

The common Method of Extracting the Square Root, Cube Root, &c. of Numbers, is only an Abridgement of the foregoing Method, and is thus perform'd, *viz.*

Place the firſt Point always over the Figure, which is in the Place of Units. Place alſo a Point over every other Figure, denoted by the Denominator of the Index of the Root to be Extracted; that is, if it be the Square Root, Cube Root, &c. that is to be Extracted, Point over the 2*d.* 3*d.* &c. Figure reſpectively to the Left Hand; and if there be any Decimals in the given Number, to the Right Hand of the ſaid Figure, which is in the place of Units. And as many Points as there are over the places of whole Numbers, ſo many places of whole Numbers muſt be in the Root, and the reſt are Decimals.

Again any Binomial as $x + y$, being Involved into it ſelf, as often as there are Units in the Denominator (1 being the Numerator) of the Index of the Root to be Extracted, produces your Canon for Evolving. Then.

1*ſt*. By the Table of Powers in Page 42 (*or otherwiſe*), find the greateſt Power that is contain'd in the firſt Period towards the Left Hand; *viz*. the greateſt Square, if it be the Square Root; the greateſt Cube, if it be the Cube-Root, *&c.* that is to be Extracted; then having plac'd the Root $= x$ in the place aſſign'd for it, which is likewiſe call'd the Root, Subſtract the ſaid Power from the ſaid Period.

After the Remainder, place the Figures in the next Period, and call that Number your Reſolvend; call alſo the Value of the Coefficient of y in the ſecond Term of your Canon, your Diviſor. Then ask how oft the Diviſor is contained in the Reſolvend omitting all the Figures in the laſt Period but the firſt, the Number $= y$ ſet in the Root next after the Value of x; then find the *Ablativum* thus. Place the Figures which are the Value of the 2*d*. Term of your Canon, ſo as the laſt of them may be under the 1*ſt* of the laſt mention'd Period, and the Values of the 3*d*. 4*th*. 5*th*. *&c.* Terms 1, 2, 3, *&c.* Places reſpectively more to the Right Hand, than thoſe of the 2*d*. Term; and the Sum of the Values of the 2*d*, 3*d*, 4*th*, 5*th*, *&c.* Terms plac'd as aforeſaid, is the ſaid *Ablativum*, which take from your Reſolvend. *But here Note*, that if the Ablativum thus found ſhould be greater than the Reſolvend, then the Value of y is too Great, and muſt be made Leſs.

After the Remainder, place the Figures in the next Period, and call that Number your Reſolvend, and call the Figures plac'd in the Root x; by which find the Value of the next y, in like manner as before directed, and ſo proceed till you have done with all your Period; and if afterwards there is a Remainder, place Cyphers after it, in order to find as many Decimal Figures as you pleaſe.

Examples.

Let it be required to Extract the Square-Root of 6968, very near.

Operation.

6968 (83.4745

$64 = xx$

Div. $16 = 2x$) 56.8 *Reſolvend.*

$48 = 2xy$

$9 = yy$

489 *Ablativum.*

Div. $166 = 2x$) 79.0.0 *Reſolvend.*

$$\begin{array}{rrl}
 & 664 = 2xy & \\
 & 16 = yy & \\
\hline
 & 6656 & \textit{Ablativum.} \\
\hline
\textit{Div.}\ 1668 = 2x) & 12440{,}0 & \textit{Reſolvend.} \\
\hline
 & 116809 = 2xy + yy & \textit{Ablativum.} \\
\hline
16694{,}4) & 075910{,}0\ ^{*} & \\
\hline
 & 667776 & \\
\hline
166948{,}5) & 0913240{,}0 & \\
\hline
 & 8347425 & \\
\hline
 & 0784975 & \textit{Remainder.} \\
\hline
\end{array}$$

* *Here I go on with the Operation as it ought to practis'd.*

The other Figures of the Root to the 11*th Fig.* Incluſive, may be found by Diviſion. Thus.

$$\begin{array}{rl}
166949{,}0) & 784975{,}0\ (47018 \\
 & 117179 \\
 & 00315 \\
 & 148 \\
\hline
 & 16 \\
\hline
\end{array}$$

Whence the Square-Root of 6968 is very near = 83.474547 018.

CHAP. III.

Evolution Mixt of Numbers, or the Method of Extracting the Roots of Affected Æquations; which Method is generally call'd the Numeral Exegeſis.

1. IF $a^3 + a$ be equal 1860990, 'tis required to find one of the Values of a.

Suppoſe $x + y = a$. Then

Canon.

$$\left.\begin{array}{ll} & a^3 = x^3 + 3x^2y + 3xy^2 + y^3 \\ \text{And} & a = x + y. \end{array}\right\} = 1860990.$$

Operation.

$1860990\ (100 + 20 + 3 = 123 = a.$

$1000000 = x^3$
$100 = x$

$1000100 = x^3 + x$

860890 *Reſolvend.*

$30000 = 3xx$
$1 = 1$

30001 *Diviſor.*

$600000 = 3x^2y$
$20 = y$ } Here $y = 20$.
$120000 = 3xyy$
$8000 = yyy$

728020 *Ablativum.*

132870 *Reſolvend.*

$43200 = 3xx$ } Here $x = 100 + 20 = 120$.
$1 = 1$

43201 *Diviſor.*

$129600 = 3x^2y$ And $y = 3$.
$3 = y$
$3240 = 3xy^2$
$27 = y^3$

132870 *Ablativum.*

0 *Remainder.*

Conſequently $100 + 20 + 3 = 123 = a$, that is, $=$ one of the Values of a.

The other affirmative Roots, if it hath any more, may be diſcover'd, by Dividing the Sum of the given Æquation, towit $a^3 + a - 1860990 = 0$ by $a - 123 = 0$, and then framing a Canon for the Quotient, as hath been already taught, and laſtly by Evolving. But I find that $a^3 + a - 1860990 = 0 \div a - 123 = 0$ is $= a^2 + 123a + 15130 = 0$.

Now the Æquation by this Quotient expreſt, to wit $a^2 + 123a = -15130$, is really an impoſſible one, and hath no Root, either Affirmative, or Negative. But it hath notwithſtanding two imaginary Roots, that is to ſay, impoſſible Roots, namely

$\sqrt{-\frac{45391}{4}} - \frac{123}{2}$ and $+\sqrt{-\frac{45391}{4}} - \frac{123}{2}$; which are found by Part X.

2*d*. Let it be required to find one of the affirmative Values of *a* in this Æquation, *viz*. $a^3 - 171.91a^2 + 7905.6a = 71460$. Suppoſe $x + y = a$. Then

Canon.

$$\left.\begin{array}{l} a^3 = x^3 + 3x^2y + 3xy^2 + y^3 \\ -171.91a^2 = -171.91x^2 - 343.82xy - 171.91y^2 \\ +7905.6a = 7905.6x + 7905.6y \\ = 71460. \end{array}\right\}$$

Firſt I ſuppoſe $a = 20$; then $a^3 - 171.91aa + 7905.6a$ $(= 8000 - 68764 + 158112) = 97348$, which is more than 71460; therefore $a \sqsupset 20$.

Again, I ſuppoſe $a = 10$; then $a^3 - 171.91aa + 7905.6a = 62865$, which is Leſs 71460; therefore $a \sqsubset 10$;

That is, a is $\sqsupset 20$, and $\sqsubset 10$; conſequently 10 is the 1ſt Member of the Root ſought; that is $10 = x$.

Operation.

$71460\ (10 + 1 + .9 + .01 = 11.91 = a.$

$$\left.\begin{array}{l} 1000 = x^3 \\ -17191 = -171.91x^2 \\ +79056 = +7905.6x \end{array}\right\} \text{Here } x = 10.$$

$62865 = x^3 - 171.91x^2 + 7905.6x$

08595 *Reſolvend.*

$$\begin{array}{l} 300 = 3xx \\ -3438.2 = -343.82x \\ +7905.6 = +7905.6 \end{array}$$

4767.4 *Diviſor.*

$$\begin{array}{l} 300 = 3x^2y. \\ -3438.2 = -343.82xy \\ +7905.6 = 7905.6y \\ +30 = 3xy^2 \\ -171.91 = -171.91y^2 \\ +1 = y^3 \end{array}$$

Here $y = 1$.

4626.49 *Ablativum.*

3968.51 *Reſolvend.*

$+ 363 = 3xx$
$- 3782.02 = - 343.82x$ } Here $x = 11$.
$+ 7905.6 = 7905.6$

4486.58 *Divisor*

$326.7 = 3x^2y$
$- 3403.818 = - 343.82xy$ } Here $y = .9$.
$+ 7115.04 = 7905.6y$
$+ 26.73 = + 3xyy$
$- 139.2471 = - 171.91yy$
$+ .729 = y^3$

3926.1339 *Ablativum.*

42.3761 *Reſolvend.*

$424.83 = 3xx$
$- 4091.458 = - 343.82x$ } Here $x = 11.9$.
$+ 7905.6 = 7905.6$

4238.972 *Diviſor.*

$4.2483 = 3x^2y$
$- 40.91458 = - 343.82xy$ } Here $y = .01$.
$+ 79.056 = 7905.6y$
$+ .00357 = 3xyy$
$- .017191 = - 171.91y^2$
$+ .000001 = y^3$

42.3761 *Ablativum.*

0 *Remainder.*

Whence $a = 11.91$.
This Æquation hath two Roots more, *viz.* 60 and 100.

CHAP. IV.

Evolution of Fractions.

FIrſt prepare the given Fraction to be Evolved, *viz.* if it be a Compound Fraction, reduce it to a Simple one (by § 4. Chap. II. Part II.); and if it be a Mixt Fraction, reduce it to an Improper one, (by § 2. Chap. II. Part II.) Then

Sect. 1. Evolution of Fractional Quantities.

If the Numerator and Denominator have each of them ſuch a Root as is required; Evolve each of them, and their reſpective Roots will be the Numerator and Denominator required.

Examples.

1. Thus the Square-Root of $\frac{1}{a}$ of $\frac{g^2}{b}$ of $\frac{a}{b} = \frac{g^2}{b^2}$ is $= \frac{g}{b}$.

2. Alſo the Cube-Root of $1 + \frac{6a^2b + 2b^3}{a^3 - 3a^2b + 3ab^2 - b^3}$ $= \frac{a^3 + 3a^2b + 3ab^2 + b^3}{a^3 - 3a^2b + 3ab^2 - b^3}$ will be found $= \frac{a+b}{a-b}$.

Sect. 2. Evolution of Fractional Numbers.

If the Fractional Numbers given can be reduc'd to Decimals of the ſame exact Value; reduce them to ſuch, and then Evolve.

Examples.

1. Thus the Cube Root of $\frac{1296}{750}$ is found to be $= 1.2$; by reducing $\frac{1296}{750}$ to a Decimal Fraction of the ſame Value; (which is done by Dividing the Numerator, with a ſufficient number of Decimal Cyphers, plac'd after it by the Denominator, till nothing remains) and then Extracting the Cube Root of that Decimal Fraction; that is, of 1.728

2. Alſo if $3y^3 - 2y^2 + 144y = 71\frac{7}{8}$; that is $= 71.875$; then one of the Values of y is $= .5$.

3 Again, If $a^5 + \frac{3}{4}a^3 = 5157\frac{5}{8}$; 'tis required to find one of the Values of a.

First reduce the Fraction in this Æquation to Decimals of the Value, and the ſaid Æquation will become $a^5 + .75a^3 = 5157.625$.

2*d*. Suppoſe $a = x + y$. Then

$$\left.\begin{aligned} a^5 &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5 \\ .75a^3 &= .75x^3 + 2.25x^2y + 2.25xy^2 + .75y^3 \end{aligned}\right\} = 5157.625.$$

Operation.

$5157.625\ (5 + .5 = 5.5 = a$ *Anſwer*.

$3125 = x^5$ Here $x = 5$.

$93.75 = .75x^3$

$3218.75 = x^5 + .75x^3$

1938.875 *Reſolvend.*

$3125 = 5x^4$

$56.25 = 2.25x^2$

3181.25 *Diviſor.*

$$\left.\begin{array}{r} 1562.5 = 5x^4y \\ 28.125 = 2.25x^2y \\ 312.5 = 10x^3y^2 \\ 2.8125 = 2.25xy^2 \\ 31.25 = 10xxy^3 \\ .09375 = .75y^3 \\ 1.5625 = 5xy^4 \\ .03125 = y^5 \end{array}\right\} y = .5.$$

1938.875 *Ablativum.*

0 *Remainder.*

But if the Fractions in the given Æquation cannot be redue'd to Decimals of the ſame Value; reduce the given Æquation to a common Denominator, and then Multiply each part of that Æquation by the common Denominator, &c.

1. Thus if $a^3 - 6\frac{109}{144}a = 2\frac{31}{144}$; 'tis required to find one of the Values of a.

The foregoing Æquation being reduc'd to a common Denominator is $\frac{144a^3 - 973a}{144} = \frac{319}{144}$.

And by Multiplying each Part by 144, we have $144a^3 - 973a = 319$.

Now ſuppoſe $a = x + y$. Then

$$\left.\begin{array}{l} 144a^3 = 144x^3 + 432x^2y + 432xy^2 + 144y^3 \\ -973a = -973x - 973y \end{array}\right\} = 319.$$

Operation.

$319\ (2 + .7 + .05 = 2.75 = a$ *Anſwer.*

5152

$1152 = 144x^3$
$-1946 = -973x$

$-794 = 144x^3 - 973x$

1113 *Reſolvend.*

$1728 = 432x^2$
$-973 = -973$

755 *Diviſor.*

$$\left.\begin{array}{r} 1209.6 = 432x^2y \\ -681.1 = -973y \\ 423.36 = 432xy^2 \\ 49.392 = 144y^3 \end{array}\right\} \text{Here } y = .7$$

1001.252 *Ablativum.*

111.748 *Reſolvend.*

$3149.28 = 432x^2$
$-973 = -973$ Here $x = 2.7$.

2176.28 *Diviſor.*

$$\left.\begin{array}{r} 157.464 = 432x^2y \\ -48.65 = -973y \\ 2.916 = 432xy^2 \\ .018 = 144y^3 \end{array}\right\} \text{Here } y = .05.$$

111.748 *Ablativum.*

0 *Remainder.*

Here you may obſerve, that by ſuppoſing $a = 2$, $144a^3 - 973a$ *is* = *a Negative Number; and conſequently the Reſolvend is greater than the given abſolute Number. But if you had ſuppos'd* $a = 3$, *then* $144a^3 - 973a$ *wou'd be* $\sqsubset$ *given abſolute Number; and therefore* x *muſt not be ſuppos'd* $= 3$.

2. And if $a^4 + a = 126\frac{64}{81}$; then $81a^4 + 81a = 10270$, and one of the Values of a is $3.33333 + = 3\frac{1}{3}$.

N.B. *Tho' I have not hitherto in this Part IV, in Dividing any Reſolvend by its reſpective Diviſor, purſued ſuch Diviſion beyond finding the firſt Significant Figure of the Quotient, or of the Value of* y. *Yet when the Diviſor once takes place, it may be continued to as many, or almoſt as many Figures, as the next preceeding* x *hath of the firſt Figures of the Root ſought; as in the following Examples.*

Example 1.

If $a^3 = 231$; Quære *a proxime.*
Suppoſe $x + y = a$; then

$(a^3 =)\ x^3 + 3x^2y + 3xy^2 + y^3 = 231.$

231 (6
$216 = x^3$

15 *Reſolvend.*

Diviſor $108 = 3xx$

$14.04 = 3x^2y$
$.3042 = 3xy^2$
$.002197 = y^3$
} Here $y = .13$.

14.346397 *Ablativum.*

.653603 *Reſolvend.*

Diviſor $112.7307 = 3x^2$

$.652710753 = 3x^2y$
$....6165082 - = 3xy^2$
$.......1941 + = y^3$
} Here $x = 6.13$. And $y = .00579$.

.6533274553 + *Ablativum.*

.0002755447 − *Reſolvend.*

Diviſor $112.9437 + = 3x^2$. Here $x = 6.13579$, and (the next $y = .00000243966$.

Wherefore a is near $= 6.13579243966$.

Example 2.

If $a^4 - 10a = 10000$; Quære a *proxime.*
Suppoſe $x + y = a$; then
$(a^4 =)\ x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
$(-10a =) -10x - 10y$
} $= 10000$.

Operation.

10000 (10

$10000 = x^4$
$-\ 100 = -10x$

$9900 = x^4 - 10x$

100 *Reſolvend.*

$4000 = 4x^3$
$-\ 10 = -10$

3990

3990 *Diviſor.*

$100 = 4x^3y$

$-\ .25 = -10y$

$.375 = 6x^2y^2$

$\ldots .625 = 4xy^3$

$\ldots\ldots .39+ = y^4$

Here $y = .025$.

$100.12562539+$ *Ablativum.*

$-\ .12562539+$ *Reſolvend.*

$4030.075+ = 4x^3$

$-\ 10 = -10$

$4020.075+$ *Diviſ.*

Here $x = 10.025$. And the next $y = -.000031249+$.

Wherefore a is near $= 10.02496875\dot{0}$.

Scholium.

Tho' the Method uſ'd in theſe two laſt Examples is better (as being more Expeditious) than that uſ'd in the foregoing Examples; yet ſtill a better Method (which is the ſame with Mr. *Raphſon's*, and by him call'd the Converging Series) may be deduc'd from this; thus in Extracting the Cube-Root of 231, in *Ex.* 1. You may readily ſee that each Diviſor is $= 3x^2$, and that each Reſolvend is $= 231 - x^3$, and conſequently, from the time the Diviſor takes place, or ſerves to diſcover the firſt Figure of the true Value of y, each $y = \frac{231 - x^3}{3x^2}$; and therefore the Theorem for Extracting the Cube-Root of any Number $= A$ is $y = \frac{A - x^3}{3x^2}$. In like manner in Example 2. *viz.* $a^4 ** - 10a = 10000$, each Diviſor is $= 4x^3 - 10$, each Reſolvend $= 10000 - x^4 + 10x$; and from the time the Diviſor takes place, each $y = \frac{10000 - x^4 + 10x}{4x^3 - 10}$, and therefore the Theorem for finding the Value of a in this Æquation, *viz.* $a^4 - pa = q$ is $y = \frac{q - x^4 + px}{4x^3 - p}$. And in this or the like Manner Theorems may be raiſ'd for Evolving all poſſible Æquations.

This is the moſt natural way of raiſing Raphſon's *Theorems for Extracting the Roots of Æquations.*

Wherefore in Extracting any Root of any Æquation, whether Simple or Affected, Let x be taken = the Value of the 1*ſt.* Figure, or 1*ſt* and 2*d* Figures of the Root ſought, then, by its reſpective Theorem, find the Value of the firſt y to ſo many Figures as you think to be the true ones; the Sum of which, and that of x call x (*i. e.* your next x), with which proceed, by its reſpective Theorem, to find the Value of the next y, and ſo on.

Thus, if the Cube-Root of 231 is to be Extracted by this Method, let 1ſt x be taken $= 6$; then 1ſt $y\left(= \frac{A - x^3}{3x^2}\right) = \frac{231 - 216}{108} = .13$; therefore $6 + .13$ (= 1ſt x + 1ſt y) $= 6.13 =$ 2d x, then 2d $y = \frac{231 - 230.346397}{112.7307} = .00579$; therefore 6.13579 (= 2d x + 2d y) = 3d x.

Then 3d $y = \frac{231 - 230.9997244553 +}{112.9437 +} = .00000243966$; therefore 6.13579243966 (= 3d x + 3d y) is very near = Cube-Root of 231.

You may ſee by this Example, *and by as many more as you are pleas'd to try, that from the time the Diviſor takes place, each Renewal doubles entirely, or almoſt, the true Figures in the laſt x; and of conſequence a few renewals afterwards will ſerve to Extract any Surd Root required to very many places of Figures.*

PART V.

Of the Indices, or Exponents of Powers.

IF an Unit be Multiplied by any Quantity *a*, and the Product *a* by *a*, and that Product *aa* by *a*, and that Product *aaa* by *a*, &c. the ſeveral Products are the 1ſt. 2d. 3d. 4th, &c. Powers of *a*.

The Antients Manner of Writing the Root, and ſeveral Powers of *a* was as follows.

	Root or 1ſt. Power,	Square or 2d. Power,	Cube or 3d. Power,	Biquadrat, or 4th Power.	&c. of *a*.
Thus	$\sqrt{a}$,	*aa* ,	*aaa* ,	*aaaa* ,	&c.
Or thus	*a* ,	*aq*: ,	*ac*: ,	*aqq*: ,	&c.

But *Des-Cartes* choſe to deſign the ſaid Powers of *a* in the following Manner, (which, for many Reaſons, is the beſt). *viz.*

Root	Square.	Cube.	4th Power,	&c. of *a*.
a, or a^1	a^2	a^3	a^4 ,	&c.

The Figures 1, 2, 3, 4, &c. writ over *a*, and ſhewing its Powers, be call'd Indices or Exponents.

It is Evident that the Powers of *a* in (or the Terms of) the foregoing Series, are in a continued Geometrical Proportion, whoſe Ratio is the 1ſt. Power or Root *a*, and the Indices of the ſaid Powers in a continued Arithmetical Proportion, whoſe common exceſs is 1 (by the Definitions of both Proportions).

Now ſince the Exponent of each Power of the foregoing Series is, by the common exceſs of the Indices, to wit by 1, more than the Index of the next foregoing Power, it muſt follow of courſe, if the ſaid Series be continued backward, that the ſeveral Terms of it will be found (by Subſtracting the ſaid common Exceſs of the Indices, to wit 1 from the Index 1 of the Root a^1, and from the Remainder 0 the ſaid common Exceſs, and from that Remainder — 1, the ſaid common Exceſs, and from that Remainder — 2, the ſaid common Exceſs, &c.) $=$ &c, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, a^4, &c.

And ſince each Power of the ſaid foregoing Series is the Product of the next foregoing Power, and of the common Ratio; of Conſequence, if the ſaid Series be continued backward, the ſeveral Terms of it will be found (by Dividing the Root a by the Ratio a; and the Quotient 1 by the ſaid Ratio; and that Quotient $\frac{1}{a}$ by the ſaid Ratio, and that Quotient $\frac{1}{a^2}$ by the ſaid Ratio, &c.) $=$ &c, $\frac{1}{a^3}$, $\frac{1}{a^2}$, $\frac{1}{a}$, 1, a, a^2, a^3, &c.

Hence 'tis manifeſt that a^0 is $= 1$, alſo $a^{-1} = \frac{1}{a}$, alſo $a^{-2} = \frac{1}{a^2}$, alſo $a^{-3} = \frac{1}{a^3}$, &c. And that the ſaid Series being continued backward and forward, may be writ

Thus &c. $a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3,$ &c.

Or thus &c, $\frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a}, 1, a^1, a^2, a^3,$ &c.

From what has been ſaid, may be deduc'd the two Fundamental Rules for all Operations relating to the Exponents of Powers. The firſt of which is that the Exponent of the Product of any two Terms in the ſaid Series in ∺ is equal to the Sum of the Exponents of the ſaid two Terms.

And the 2*d.* is, that the Exponent of the Quotient of any two Terms in the ſaid Series in ∺ is had by Subſtracting the Exponent of the Diviſor from the Dividends Exponent.

For the 1*ſt.* Rule may be prov'd by *Algebraical* Multiplication, by the help of what has been already ſaid in this Part V. So the Exponent of the Product of a^6 and a^4 is prov'd to be $= 6 + 4 = 10$, by Multiplying a^6 by a^4; for $a^6 \times a^4$ is $= a^{10} = a^{6+4}$. In like manner the Exponent of the Product of a^{-4} and a^3 may be prov'd to be $= -4 + 3 = -1$; thus a^{-4} is (by what has been prov'd in this Part V.) $= \frac{1}{a^4}$, and $\frac{1}{a^4} \times a^3 = \frac{a^3}{a^4} = \frac{a^3}{a^3} \div a = 1 \div a = \frac{1}{a}$; But $\frac{1}{a} = a^{-1}$; conſequently -1 is the Exponent of $a^{-4} \times a^3$.

And the 2d. Rule muſt follow of Courſe from the 1ſt, or may be prov'd by *Algebraical* Diviſion; ſo the Exponent of $a^6 \div a^4$ is prov'd to be $= 6 - 4 = 2$, by Dividing a^6 by a^4; for $a^6 \div a^4 = \frac{a^6}{a^4} = a^2$. Alſo the Exponent of the Quotient of a^3 Divided by a^5 is prov'd to be $= 3 - 5 = -2$; thus $\frac{a^3}{a^5} = \frac{a^3}{a^3} \div a^2 = \frac{1}{a^2} = a^{-2}$ (by what has been prov'd in this Part V.) In like manner the Exponent of the Quotient of a^{-4} Divided by a^{-3} is prov'd to be $= -4 + 3 = -1$, thus a^{-4} (by what has been ſaid in this Part) is $= \frac{1}{a^4}$, and $a^{-3} = \frac{1}{a^3}$; But $\frac{1}{a^4} \div \frac{1}{a^3}$ is $= \frac{a^3}{a^4} = \frac{a^3}{a^3} \div a = 1 \div a = \frac{1}{a}$; that is $= a^{-1}$; &c.

Or I may prove that as a^1, a^2, a^3, &c, is a Series of Terms in ∺ increaſing; ſo a^2, a^1, a^0, a^{-1}, a^{-2}, a^{-3}, &c. is a Series of Terms in ∺ Decreaſing; thus,

Suppoſe the ſaid Series of Terms in ∺ Decreaſing to be $= a^2$, a^1, a^u, a^w, a^x, a^y, &c. reſpectively; then a^2, a^1, a^u, are in ∺ that is $a^2 \,..\, a^1 :: a^1 \,..\, a^u$.

But the Product of the Means is equal to the Product of the Extreams; that is $a^1 \times a^1 = a^2 \times a^u$; therefore (by what has been already ſaid in this Part V.) $1 + 1 = 2 + u$; conſequently $u = 0$. Again, ſince a^1, a^0, a^w, are in ∺; then $a^1 \,..\, a^0 :: a^0 \,..\, a^w$; therefore $0 + 0 = 1 + w$, and $-1 = w$. In like manner x will be prov'd $= -2$; as alſo $y = -3$, &c.

The manner of raiſing any given Quantity to any required Power, or Extracting any Root out of it, is eaſily deducible from the two foregoing Rules; thus the Square of a^3 is $= a^3 \times a^3 = a^{3+3} = a^6$, and $6 = 3 \times 2$; that is, the Exponent of the Square is double the Exponent of the Root; or if you ſuppoſe a^6 to be the Root, then a^3 will be the Square-Root; wherefore (ſince $6 \div 3 = 2$) you may conclude that the Exponent of the Square-Root is half the Index of the Root. Alſo the Cube of $y^3 = y^3 \times y^3 \times y^3 = y^{3+3+3} = y^9$, and $9 = 3 \times 3$, which ſhews that the Exponent of the Cube is triple the Exponent of the Root, or that the Exponent of the Cube-Root is one third Part of the Exponent of the Root. Alſo $y^{-1} \times y^{-1} \times y^{-1}$ is $= y^{-1-1-1} = y^{-3}$. And univerſally the Exponent of the m Power, is m times the Exponent of the Root, and the Exponent of the m-Root (or $\frac{1}{m}$ Power) is $\frac{1}{m}$ times the Exponent of the Root.

Hence the m-Power of the m-Root is $=$ the Root; that is the Square of the Square-Root, or the Cube of the Cube-Root, &c. is equal to the Root.

By the laſt Paragraph but one it is evident that $\sqrt[2]{a}, \sqrt[3]{a}, \sqrt[4]{a}$, &c. are $= a^{\frac{1}{2}}, a^{\frac{1}{3}}, a^{\frac{1}{4}}$, &c. reſpectively; for a^1 being the Root, half its Index, towit the half of 1 muſt be the Index of the Square Root; alſo $\frac{1}{3}$ of the Index of the Root muſt be the Index of the Cube-Root; &c.

☞ The Index of the Square Root, Cube-Root, &c. may be otherwiſe diſcover'd. Thus

Let it be required to find the Square-Root of a^3.

Suppoſe it (to wit the Square-Root of a^3) $= a^u$.

Then $a^u \times a^u = a^3$; that is, the Square of the Square-Root is = the Root; wherefore (by 1ſt Rule) $u + u = 3$, and $u = \frac{3}{2}$, conſequently $a^{\frac{3}{2}}$ is the Square Root of a^3

Again, let it be required to find the Cube-Root of a^{-2}.

Suppoſe it $= a^w$, then $a^w \times a^w \times a^w = a^{3w}$ (by the 1ſt Rule) is $= a^{-2}$; wherefore $3w = -2$ and $w = -\frac{2}{3}$. Conſequently $a^{-\frac{2}{3}}$ is the Cube Root of a^{-2}. &c.

So that this Remark is only a Confirmation of the foregoing univerſal Rule.

The Sum of what has been ſaid in relation to the Indices or Exponents of Powers is,

Putting $a^m =$ any Quantity whatſoever, and $a^n =$ the ſame or any other Quantity.

$$\left.\begin{array}{ll}\text{That 1ſt.} & a^m \times a^n \\ \text{2d.} & a^m \div a^n \\ \text{3d.} & a^m \text{ rais'd to the } n \text{ Power} \\ \text{4th.} & \text{the } n \text{ Root of } a^m\end{array}\right\} \text{is} \left\{\begin{array}{l} = a^{m+n}. \\ = a^{m-n}. \\ = a^{mn}. \\ = a^{\frac{m}{n}}. \end{array}\right.$$

Here follow more Varieties, which in reality are included in the 4 laſt foregoing Articles, ſince m and n are Univerſal, and conſequently either or both of them may be equal to any Quantities Whole or Fracted, Affirmative or Negative.

$$\left.\begin{array}{l} a^{-q} \times a^{+p} = a^{-q+p} \\ a^{-q} \div a^{p} = a^{-q-p} \\ a^{p} \div a^{-q} = a^{p+q} \\ a^{-q} \text{ rais'd to } p \text{ Power} = a^{-qp} \end{array}\right\} \begin{array}{l} a^{-q} \times a^{-r} = a^{-q-r} \\ a^{-q} \div a^{-r} = a^{-q+r} \\ \text{the } r \text{ Root of } a^{-q} \text{ is } = a^{\frac{-q}{r}} \end{array}$$

Again $a^{-\frac{r}{t}} \times a^{\frac{s}{w}} = a^{-\frac{r}{t}+\frac{s}{w}} = \overline{a|}^{\frac{-rw+ts}{tw}}$

$a^{-\frac{r}{t}} \div a^{\frac{s}{w}} = a^{-\frac{r}{t}-\frac{s}{w}} = \overline{a|}^{\frac{-rw-ts}{tw}}$

$a^{\frac{s}{w}} \div a^{-\frac{r}{t}} = a^{\frac{s}{w}+\frac{r}{t}} = \overline{a|}^{\frac{st+rw}{tw}}$

$a^{-\frac{r}{t}}$ rais'd to $\frac{s}{w}$ Power $= \overline{a|}^{-\frac{rs}{tw}}$

$a^{\frac{s}{w}}$ rais'd to $\frac{r}{t}$ Power $= \overline{a|}^{\frac{sr}{tw}}$

The $\frac{s}{w}$ Root of $\overline{a|}^{-\frac{r}{t}} = a^{-\frac{r}{t}\div\frac{s}{w}} = a^{-\frac{wr}{st}}$

The $\frac{r}{t}$ Root of $a^{\frac{s}{w}} = a^{\frac{s}{w}\div\frac{r}{t}} = \overline{a|}^{\frac{ts}{wr}}$

Alſo $a^{-\frac{r}{t}} \times a^{-\frac{s}{w}} = a^{-\frac{r}{t}-\frac{s}{w}}$

And $a^{-\frac{r}{t}} \div a^{-\frac{s}{w}} = a^{-\frac{r}{t}+\frac{s}{w}}$

PART

PART VI.

Of Surd-Roots.

CHAP. I.

Definitions.

WHen any Number or Quantity hath its Root propos'd to be Extracted, and yet is not a true figurat Number of that kind; that is, if its Square Root being demanded, it ſelf is not a true Square; if its Cube-Root being required, it ſelf be not a true Cube; &c. that Root is called a Surd Root, becauſe it is inexpreſſible by any known way of Notation, otherwiſe than by its Index or Radical Sign; ſo the Square-Root of 2 can be only writ thus $\overline{2}|^{\frac{1}{2}}$ or thus $\sqrt{2}$; for 'tis evident that the Square-Root of 2 is not a whole Number; neither is it a Fraction, becauſe the Square of a Fraction is alſo a Fraction, (by the 16. 8. *Eucl. El.*) and conſequently it is not expreſſible by any rational Number. Alſo the Cube-Root of b^2 can be only writ thus $\overline{b^2}|^{\frac{1}{3}} (= b^{\frac{2}{3}})$ or thus $\sqrt[3]{b^2}$. &c.

2. But altho theſe Surd-Roots (when truly ſuch) are inexpreſſible by rational Numbers, they are notwithſtanding capable of *Algebraical* Operations, as ſhall be hereafter ſhew'd.

Surds are either Simple, which are expreſs'd by one ſingle Term as $b^{\frac{1}{2}}$, $\sqrt[3]{c}$, $\overline{bd}|^{\frac{2}{4}}$, &c; or Compound, which are form'd by the Addition and Subſtraction of Simple Surds, as $\sqrt{5} + \sqrt{2}$, $5^{\frac{1}{2}} - 2^{\frac{1}{2}}$, $d^{\frac{1}{2}} + b^{\frac{1}{3}} - c^{\frac{1}{4}}$, &c. or elſe univerſal as $\overline{7 + 2^{\frac{1}{2}}}|^{\frac{1}{3}}$ which ſignifies the Cubick-Root of that Number which is the reſult of adding 7 to the Square-Root of 2, &c.

3. Surds are alſo Commenſurable or Incommenſurable.

Commenſurable Surd-Roots are ſuch, whoſe reaſon or proportion to one another, may be expreſt by rational Numbers or Quantities.

And ſuch Surd-Roots whoſe proportion cannot be expreſt by rational Numbers or Quantities are call'd Incommenſurable.

CHAP.

CHAP. II.

Reduction of Simple Surds.

Propoſition 1.

TO reduce Simple Heterogeneal Surds; that is, ſuch as are under different Indices, to the leaſt Homogeneals, or to ſuch as may have the leaſt common Index.

Rule.

Reduce the Indices of the given Surds to a common Denominator, then place the greateſt common Meaſure of the Numerators as a Numerator over the common Denominator, which Fraction, when reduc'd to its loweſt Terms, is the common Index ſought: then Divide each of the ſaid Indices by the ſaid common Index, and then Involve each given Number or Quantity without its Index, according to the Index denoted by its reſpective Quotient, and over each Power thus produc'd, place the ſaid common Index, and you will have the Surds required.

Examples.

Reduce $\overline{12}|^{\frac{1}{4}}$ and $\overline{7}|^{\frac{1}{6}}$ to two other Surds of the ſame Value that will have a common Index.

$\frac{1}{4}$ and $\frac{1}{6}$ reduc'd to a common Denominator, are equal to $\frac{6}{24}$ and $\frac{4}{24}$.

The greateſt common Meaſure of the Numerators 6 and 4 is 2.

Wherefore $\frac{2}{24}$ when reduc'd to its ſmalleſt Terms, that is, $\frac{1}{12}$ is the common Index ſought, by which the 1*ſt* ($\frac{1}{4}$) and 2*d.* ($\frac{1}{6}$) Indices being Divided, the Quotients are 3 and 2 reſpectively.

The 1*ſt* of the given Numbers without its Index, that is 12 Involved according to the Index (3) denoted by the 1*ſt* Quotient, is $12^3 = 1728$.

Alſo the 2*d* given Number without its Index, to wit 7 Involved according to the Index (2) denoted by the 2*d* Quotient, is $7^2 = 49$.

Conſequently $\overline{1728}|^{\frac{1}{12}}$ and $\overline{49}|^{\frac{1}{12}}$ are the Surds required reſpectively.

2. Reduce $\overline{b}|^{\frac{2}{3}}$ and $\overline{dc}|^{\frac{2}{9}}$ to two other Surds of the ſame Value that ſhall have a common Index.

$\frac{2}{3}$ and $\frac{2}{9}$ reduc'd to a common Denominator are equal to $\frac{18}{27}$ and $\frac{6}{27}$ reſpectively.

The greateſt common Meaſure of the Numerators is 6.

Wherefore $\frac{2}{9}(=\frac{6}{27})$ is the common Index ſought.

By which the 1ſt ($\frac{2}{3}$) and the 2d ($\frac{2}{9}$) Indices being Divided, the 1ſt Quotient will be 3, and the 2d Quotient 1.

The 1ſt of the given Quantities without its Index, to wit b Involv'd according to the Index (3) denoted by the ſaid 1ſt Quotient is b^3.

Alſo the 2d given Quantity, without its Index, that is, dc Involved according to the Index (1) denoted by the ſaid 2d Quotient, is cd; wherefore $\overline{b^3}|^{\frac{2}{9}}$ and $\overline{cd}|^{\frac{2}{9}}$ are the Surds required.

3. Let it be required to reduce $\overline{a}|^{\frac{i}{n}}$ and $\overline{e}|^{\frac{t}{m}}$ to two other Surds of the ſame Value that ſhall have a common Index.

$\frac{i}{n}$ and $\frac{t}{m}$ reduc'd to a common Denominator are equal to $\frac{mi}{mn}$ and $\frac{nt}{mn}$ reſpectively; And the greateſt (known) common Meaſure of the Numerators mi and nt is 1; wherefore (by our Rule) $\frac{1}{mn}$ is the common Index ſought; By which the 1ſt ($\frac{i}{n}$) and 2d ($\frac{t}{m}$) Indices being Divided, the 1ſt and 2d Quotients are mi and nt reſpectively. The 1ſt given Quantity without its Index, that is a Involv'd according to the Index (mi) denoted by the firſt Quotient is a^{mi}; and the 2d given Quantity without its Index, that is e Involved according to the Index (nt), denoted by the 2d Quotient, is e^{nt}; Wherefore (by our Rule) $\overline{a^{mi}}|^{\frac{1}{mn}}$ and $\overline{e^{nt}}|^{\frac{1}{mn}}$ are the Surds required.

Here you have three Examples at large of our Rule, the laſt of which, being Univerſal, may ſerve to Demonſtrate it.

For $\overline{a^{mi}}|^{\frac{1}{mn}}$ is $= a^{\frac{mi}{mn}}$ which is manifeſtly $= a^{\frac{i}{n}}$.

And $\overline{e^{nt}}|^{\frac{1}{mn}}$ is $= e^{\frac{nt}{mn}}$, which is manifeſtly $= e^{\frac{t}{m}}$.

Corollary.

Hence rational Quantities may be reduc'd to the form of Surds, which ſhall have any aſſigned Indices.

Thus, if it was required to reduce a to the form of a Surd, whoſe Index will be $\frac{1}{n}$.

Firſt Divide the Index of a, to wit 1 by $\frac{1}{n}$ and the Quotient is n. Then a Involv'd into itſelf according to the Index denoted by n, gives a^n;

Conſequently $\overline{a^n}|^{\frac{1}{n}}$ is the Quantity required, which manifeſtly is $= a$.

Propoſition 2.

To reduce Surd-Roots to their moſt ſimple Terms.

Rule.

1*ſt*. Find the greateſt Power, which being affected with the Index (or Radical Sign) of the given Surd, ſhall have a rational Root, and which will Divide the given Number or Quantity without its Index, ſo as to leave no Remainder. By which Power Divide the ſaid Number or Quantity without its Index, and to the Quotient affected with the ſaid Index, prefix the ſaid rational Root with the Sign ×, ſo you have a new Surd equal to the propos'd one, and in more Simple Terms.

☞ But when ſuch a Power, by which the Diviſion neceſſary to ſuch Contraction is to be perform'd, cannot be readily diſcern'd, ſearch out all the Numbers or Quantities (by Lemma to Chap. V. Part XII.) which will divide the given Number or Quantity, omitting its Index (or Radical Sign) without a Remainder; and the greateſt of thoſe Diviſors, which, being affected with the Index of the given Surd, hath a rational Root, is the greateſt Power by which the Diviſion in manner aforeſaid, is to be perform'd.

Examples.

1. So inſtead of $\surd 63$, you may write $3\surd 7$; for 9 is the greateſt Square that can be had in 63, which will Divide it without a Remainder. And $63 \div 9 = 7$; wherefore (by our Rule) $\surd 63$, in its ſimpleſt Terms is $= 3 \times \surd 7 = 3 \times \overline{7}|^{\frac{1}{2}}$.

2. Alſo if it was required to reduce $\overline{a^3b + a^2b^2}|^{\frac{1}{2}}$ to its ſimpleſt Terms, it may be done thus; *viz.*

a^2 is the greateſt Power, which being affected with the Index $\frac{1}{2}$ will have a rational Root, and which will Divide $a^3b + a^2b^2$ without a Remainder.

And $\overline{a^3b + a^2b^2} \div aa = ab + b^2$: Conſequently $\overline{a^3b + a^2b^2}|^{\frac{1}{2}}$ in its ſimpleſt Terms is $= a \times \overline{ab + bb}|^{\frac{1}{2}} = a\sqrt{ab + b^2}$

3. And Univerſally $\overline{a^n b}|^{\frac{1}{n}}$ reduc'd to its ſimpleſt Terms is $= a \times \overline{b}|^{\frac{1}{n}}$. For a^n is the greateſt Power, which being affected with the Index $\frac{1}{n}$ will have a rational Root, and by which $a^n b$ being Divided, will leave no Remainder. And $a^n b \div a^n = b$; wherefore (by our Rule) $\overline{a^n b}|^{\frac{1}{n}}$, being reduc'd to its moſt ſimple Terms, is $= a \times \overline{b}|^{\frac{1}{n}} = a \sqrt[n]{b}$.

This laſt Example is an Univerſal one, by which the foregoing Rule may be Demonſtrated thus.

$\overline{a^n}|^{\frac{1}{n}}$ is $= a$.

And $\overline{a^n b}|^{\frac{1}{n}} = \overline{a^n}|^{\frac{1}{n}} \times \overline{b}|^{\frac{1}{n}}$

Wherefore $\overline{a^n b}|^{\frac{1}{n}} = a \times \overline{b}|^{\frac{1}{n}}$.

And that it is in the leaſt Terms that it can be reduc'd to, is certain, if the Diviſor be the greateſt Power that can be had in it. Q. E. D.

Propoſition 3.

To find whether two Surd-Roots given, are Commenſurable or not.

Rule.

If they have different Indices, reduce them to their leaſt common Index (by the *1ſt* Rule). And if then they are Fractions, reduce them to a common Denominator (by Part II. Chap. II. *Sect.* 2.) then

Divide (*or Multiply*) them ſeverally by ſuch a Number or Quantity, as will give one Quotient (*or Product*) Rational; then if the other Quotient (*or Product*) be Rational too, the Surds are Commenſurable, and in proportion to each other, as the ſaid Quotients (*or Products*). But if the other Quotient (*or Product*) be not Rational, the given Surds are Incommenſurable.

N. B. The Diviſor commonly us'd in this Caſe, is the greateſt Common one, which will give Rational Quotients, if the given Surds be Commenſurable.

Examples.

Thus, if it be required to find whether $\overline{12}|^{\frac{1}{2}}$ and $\overline{3}|^{\frac{1}{2}}$ are Commenſurable, and in what proportion to one another, it may be done thus;

The greateſt common Meaſure of $\sqrt{12}$ and $\sqrt{3}$ is $\sqrt{3}$; by which $\overline{12}|^{\frac{1}{2}}$ and $\overline{3}|^{\frac{1}{2}}$ being Divided, the reſpective Quotients are $\overline{4}|^{\frac{1}{2}}$ and $\overline{1}|^{\frac{1}{2}}$ that is 2 and 1; wherefore $\overline{12}|^{\frac{1}{2}}$ and $\overline{3}|^{\frac{1}{2}}$ are (by our Rule) Commenſurable, and in proportion as 2 to 1.

That is $\overline{12}|^{\frac{1}{2}} .. \overline{3}|^{\frac{1}{2}} :: 2 .. 1$.

2. Let it be required to find whether $\overline{\frac{b}{c}}|^{\frac{1}{2}}$ and $\overline{\frac{c^6}{b^6}}|^{\frac{1}{4}}$ be Commenſurable, and in what proportion they are to one another.

$\overline{\frac{b}{c}}|^{\frac{1}{2}}$ and $\overline{\overline{\frac{c}{b}}|^{6}}|^{\frac{1}{4}}$ being reduc'd to their leaſt common Index (without conſidering that $\frac{c^6}{b^6}$ may be Depreſs'd lower) is $\overline{\frac{bb}{cc}}|^{\frac{1}{4}}$, and $\overline{\frac{c^6}{b^6}}|^{\frac{1}{4}}$; and theſe reduc'd to a common Denominator are equal to $\overline{\frac{b^8}{c^2b^6}}|^{\frac{1}{4}}$ and $\overline{\frac{c^8}{c^2b^6}}|^{\frac{1}{4}}$, which being Divided by their greateſt common Diviſor $\overline{\frac{1}{c^2b^6}}|^{\frac{1}{4}}$, the Quotients are Rational, and equal to b^2 and c^2: For the first Quotient is $\overline{b^8}|^{\frac{1}{4}} = b^2$, and the 2d is $\overline{c^8}|^{\frac{1}{4}} = c^2$.

Conſequently the two given Surds are Commenſurable to one another, and in proportion as b^2 is to c^2; that is $\overline{\frac{b}{c}}|^{\frac{1}{2}} .. \overline{\frac{c^6}{b^6}}|^{\frac{1}{4}} :: b^2 .. c^2$.

This may be Demonſtrated by Æquating the Product of the Extreams to that of the Means, thus

$\overline{\frac{b}{c}}|^{\frac{1}{2}} \times c^4 = \overline{\frac{b}{c}}|^{\frac{1}{2}} \times \overline{c^8}|^{\frac{1}{2}} = \overline{c^7 b}|^{\frac{1}{2}}$ is $=$ the Product of the Extreams.

And $\overline{\frac{c^6}{b^6}}|^{\frac{1}{4}} \times b^2 = \overline{\frac{c^6}{b^6}}|^{\frac{1}{4}} \times \overline{b^8}|^{\frac{1}{4}} = \overline{c^6\, b^2}|^{\frac{1}{4}} = \overline{c^3 b}|^{\frac{1}{2}}$ is $=$ Product of the Means.

And the former Product is $=$ the latter, which proves, in ſome meaſure, our Rule.

3. Find whether $\overline{b}|^{\frac{1}{n}}$ and $\overline{c^n b}|^{\frac{1}{n}}$ be Commenſurable, as alſo their mutual Proportion.

The greateſt Common Diviſor of $\overline{b}|^{\frac{1}{n}}$ and $\overline{c^n b}|^{\frac{1}{n}}$ is $= \overline{b}|^{\frac{1}{n}}$, by which each being Divided, the Quotients are $\overline{1}|^{\frac{1}{n}}$ and $\overline{c^n}|^{\frac{1}{n}}$; But $1^{\frac{1}{n}}$ and $\overline{c^n}|^{\frac{1}{n}}$ are Rational Quantities, and equal to 1 and c reſpectively; conſequently the given Surds are Commenſurable, and

$\overline{b}|^{\frac{1}{n}} \,..\, \overline{bc^n}|^{\frac{1}{n}} :: 1 \,..\, c.$

CHAP. III.

Multiplication of Simple Surds.

Rule.

IF the given Surds be not of the ſame kind; that is, if they have different Indices: reduce them to one and the leaſt kind; that is, to their leaſt common Index, (by the *1ſt* Propoſition of the laſt Chap.) then Multiply them by one another, without their Indices; and laſtly annex the ſaid leaſt common Index to the Product. So this new Root ſhall be the Product ſought.

Examples.

So $2^{\frac{1}{2}} \times 3^{\frac{1}{2}}$ is $= 6^{\frac{1}{2}}$

Alſo $b^{\frac{1}{4}} \times c^{\frac{1}{4}}$ is $= \overline{bc}|^{\frac{1}{4}}$

And univerſally $a^{\frac{1}{n}} \times z^{\frac{1}{n}} = \overline{az}|^{\frac{1}{n}}$

Again $\overline{2}|^{\frac{1}{2}} \times \overline{3}|^{\frac{1}{3}}$ is (by the 1*ſt Propoſition* of the laſt Chapter) $= \overline{8}|^{\frac{1}{6}} \times \overline{9}|^{\frac{1}{6}} = \overline{72}|^{\frac{1}{6}}$.

Alſo $a^{\frac{1}{3}} \times \overline{b}|^{\frac{1}{4}}$ is (by the 1*ſt Propoſition* of the laſt Chapter) $= \overline{a^4}|^{\frac{1}{12}} \times \overline{b^3}|^{\frac{1}{12}} = \overline{a^4b^3}|^{\frac{1}{12}}$

And univerſally $\overline{x}|^{\frac{1}{m}} \times \overline{y}|^{\frac{1}{n}}$ is (by the 1*ſt Propoſition* of the laſt Chapter) $= \overline{x^n}|^{\frac{1}{mn}} \times \overline{y^m}|^{\frac{1}{mn}} = \overline{x^ny^m}|^{\frac{1}{mn}}$.

Scholia.

1. When a Surd is to be Multiplyed by a rational Quantity, it will be ſometimes convenient to Connex them with the Sign ×, ſo the Product of a and $\overline{b}|^{\frac{1}{n}}$ may be writ thus $a \times \overline{b}|^{\frac{1}{n}}$ or thus, $a\,\sqrt[n]{b}$.

2. When two Rational Quantities are joyn'd with the Sign × to two Surds of the ſame kind. Multiply the Rational Part by the Rational, and the Surd Part by the Surd, and the two Products joyn'd together with the Sign × is the Product required. So $a \times \overline{b}|^{\frac{1}{n}} \times c \times \overline{d}|^{\frac{1}{n}}$ is $= ac \times \overline{bd}|^{\frac{1}{n}}$ (for $a \times \overline{b}|^{\frac{1}{n}}$ is $= \overline{a^nb}|^{\frac{1}{n}}$ and $c \times \overline{d}|^{\frac{1}{n}}$ is $= \overline{c^nd}|^{\frac{1}{n}}$; But $\overline{a^nb}|^{\frac{1}{n}} \times \overline{c^nd}|^{\frac{1}{n}} = \overline{a^nc^n\,bd}|^{\frac{1}{n}}$ which is manifeſtly $= ac \times \overline{bd}|^{\frac{1}{n}}$).

3. When any Surd is to be raiſ'd to any given Power; Multiply the Index of the given Surd by the Number that Denotes that Power.

Thus $a^{\frac{1}{n}}$ raiſ'd to the n^{th} Power is $= a^{\frac{1}{n} \times n} = a^{\frac{n}{n}} = a$; alſo $a^{\frac{1}{n}}$ raiſ'd to the m^{th} Power is $= a^{\frac{1}{n} \times m} = \overline{a}|^{\frac{m}{n}}$.

You have Examples enough of this kind, in Part V.

CHAP.

CHAP. IV.

Diviſion of Simple Surds.

Rule.

IF the Surds be of the ſame kind, Divide the Dividend by the Diviſor, without their Indices, and over the Quotient place the common Index, and this new Root is the Quotient ſought; But if they be not of the ſame kind, firſt reduce them to the ſame kind (by Chap. 2. *Propoſition* 1.) and then proceed in manner aforeſaid.

So $\sqrt{20}$ Divided by $\sqrt{5}$, gives $\sqrt{4}$ for a Quotient.

Alſo $\overline{ab}|^{\frac{1}{4}} \div \overline{ca}|^{\frac{1}{4}} = \overline{\frac{ab}{ac}}\Big|^{\frac{1}{4}} = \overline{\frac{b}{c}}\Big|^{\frac{1}{4}}$.

And univerſally $\overline{z}|^{\frac{1}{r}} \div \overline{y}|^{\frac{1}{s}}$ is (by *Propoſition* 1. Chapter 2.) $=$

$\overline{z^s}|^{\frac{1}{rs}} \div \overline{y^r}|^{\frac{1}{rs}} = \overline{\frac{z^s}{y^r}}\Big|^{\frac{1}{rs}}$

When the Dividend and Diviſor are two Rational Quantities prefixt to two Surds of the ſame kind; Divide the Rational Part of the Dividend, by the Rational Part of the Diviſor; As alſo the Surd Part of the Dividend, by the Surd Part of the Diviſor: and the two Quotients connected together with (*or ſometimes without*) the Sign $\times$ is the Quotient deſired.

So $ab\sqrt[m]{cd} \div bd\sqrt[m]{c}$ is $= \frac{a}{d}\sqrt[m]{d}$.

Alſo $ab \times \overline{cd}|^{\frac{1}{m}} \div b \times \overline{cd}|^{\frac{1}{m}}$ is $= a \times \overline{1}|^{\frac{1}{m}} = a$.

CHAP.

CHAP. V. and VI.

Addition and Subſtraction of Simple Surds.

Rule.

DIvide the given Surds by their greateſt common Diviſor, and if the Quotients be Rational (that is to ſay, if the given Surds are Commenſurable) Multiply the Sum of the ſaid Quotients by the ſaid greateſt common Diviſor, and the Product ſhall be the Sum of the Surds propos'd; or Multiply the difference of the ſaid Rational Quotients by their ſaid greateſt common Diviſor, and the Product ſhall be the difference of the two Surds propos'd.

Examples.

1. Let it be required to add $\sqrt{8}$ to $\sqrt{32}$.

Their greateſt common Diviſor is $\sqrt{8}$.

By which each of the given Surds being Divided, the Quotients are $\sqrt{1}$ and $\sqrt{4}$, which are Rational Numbers, and equal to 1 and 2.

The Sum of which Quotients is 3, which being Multiplyed by the greateſt common Diviſor ($\sqrt{8}$) the Product is $3 \times \sqrt{8} = \sqrt{9} \times \sqrt{8} = \sqrt{72}$, which is $= \sqrt{8} + \sqrt{32}$.

2. I demand the Sum of $\left.\overline{\frac{b}{c}}\right|^{\frac{1}{2}}$ and $\left.\overline{\frac{c^6}{b^6}}\right|^{\frac{1}{4}}$?

$\left.\overline{\frac{b}{c}}\right|^{\frac{1}{2}}$ and $\left.\overline{\frac{c^6}{b^6}}\right|^{\frac{1}{4}}$ reduc'd to a common Index are equal to $\left.\overline{\frac{b^2}{c^2}}\right|^{\frac{1}{4}}$ and $\left.\overline{\frac{c^6}{b^6}}\right|^{\frac{1}{4}}$.

And theſe reduc'd to a common Denominator are equal to $\left.\overline{\frac{b^8}{c^2b^6}}\right|^{\frac{1}{4}}$ and $\left.\overline{\frac{c^8}{c^2b^6}}\right|^{\frac{1}{4}}$, whoſe greateſt common Diviſor is $\left.\overline{\frac{1}{c^2b^6}}\right|^{\frac{1}{4}}$.

By which each of them being Divided, the Quotients will be $\left.\overline{b^8}\right|^{\frac{1}{4}}$ and $\left.\overline{c^8}\right|^{\frac{1}{4}}$, which are Rational Quantities, and equal to b^2 and c^2.

The Sum of which Quotients is $b^2 + c^2$, which being Multiplyed by the greateſt common Diviſor $\overline{\frac{1}{c^2b^6}}\Big|^{\frac{1}{4}}$ the Product is

$$\overline{b^2 + c^2} \times \overline{\frac{1}{c^2 + b^6}}\Big|^{\frac{1}{4}} \text{ which is} = \overline{\frac{b}{c}}\Big|^{\frac{1}{2}} + \overline{\frac{c^6}{b^6}}\Big|^{\frac{1}{4}}.$$

3. I demand the Sum of $\overline{bd^n}|^{\frac{1}{n}}$ and $\overline{bc^n}|^{\frac{1}{n}}$?

Their greateſt common Diviſor is $\overline{b}|^{\frac{1}{n}}$.

By which each of them being Divided, the Quotients are $\overline{d^n}|^{\frac{1}{n}}$ and $\overline{c^n}|^{\frac{1}{n}}$, which are Rational Quantities, and equal to d and c, The Sum of which is $d + c$; which being Multiplyed by the greateſt common Diviſor $\overline{b}|^{\frac{1}{n}}$, the Product is $\overline{d + c} \times \overline{b}|^{\frac{1}{n}}$ which is $= \overline{bd^n}|^{\frac{1}{n}} + \overline{bc^n}|^{\frac{1}{n}}$.

This laſt Example being Univerſal, will ſerve to demonſtrate our Rule.

Suppoſe $\overline{b}|^{\frac{1}{n}} = a$.

Then $\overline{bd^n}|^{\frac{1}{n}}$ is $= a \times \overline{d^n}|^{\frac{1}{n}} = ad$.

And $\overline{bc^n}|^{\frac{1}{n}}$ is $= a \times \overline{c^n}|^{\frac{1}{n}} = ac$.

But $ad + ac = \overline{d + c} \times a =$ Sum of the two given Quantities.

Subſtitute $\overline{b}|^{\frac{1}{n}}$ for a, and you'll have $\overline{d + c} \times \overline{b}|^{\frac{1}{n}} =$ Sum of the two given Quantities. *W. W. D.*

In like manner if it were required to Subſtract $\overline{b}|^{\frac{1}{n}}$ from $\overline{bc^n}|^{\frac{1}{n}}$ the Remainder will be found (by our Rule) to be $\overline{c - 1} \times \overline{b}|^{\frac{1}{n}}$.

Or if it were required to find the Value of $\overline{b}|^{\frac{1}{n}} - \overline{bc^n}|^{\frac{1}{n}}$ in other Terms, it will (by our Rule) be found $= \overline{1 - c} \times \overline{b}|^{\frac{1}{n}}$.

Wh

When the given Surds are Incommenſurable, neither their Sum nor Difference can be expreſt by any Simple Root; but they are to be added by +, and Subſtracted by —; whence ariſe Surds, Binomial and Reſidual.

Thus the Sum of $\sqrt{6}$ and $\sqrt{7}$ is $\overline{6}|^{\frac{1}{2}} + \overline{7}|^{\frac{1}{2}}$, and the difference of $6^{\frac{1}{2}}$ and $7^{\frac{1}{2}}$ is $7^{\frac{1}{2}} - 6^{\frac{1}{2}}$, &c.

But Univerſally the Sum of $\overline{b}|^{\frac{1}{n}}$ and $\overline{c}|^{\frac{1}{n}}$ is (manifeſtly) $= \overline{\overline{b^{\frac{1}{n}} + c^{\frac{1}{n}}}|^{n}}|^{\frac{1}{n}}$, or the difference of $\overline{b}|^{\frac{1}{n}}$ and $\overline{c}|^{\frac{1}{n}}$ is $= \overline{\overline{b^{\frac{1}{n}} \backsim c^{\frac{1}{n}}}|^{n}}|^{\frac{1}{n}}$; Wherefore $\overline{b}|^{\frac{1}{2}} + c^{\frac{1}{2}}$ is $= \overline{\overline{b^{\frac{1}{2}} + c^{\frac{1}{2}}}|^{2}}|^{\frac{1}{2}} = \overline{b + c + 2\sqrt{bc}}|^{\frac{1}{2}}$,* and $b^{\frac{1}{3}} + c^{\frac{1}{3}}$ is $= \overline{\overline{b^{\frac{1}{3}} + c^{\frac{1}{3}}}|^{3}}|^{\frac{1}{3}} = \overline{b + c + 3b^{\frac{2}{3}}c^{\frac{1}{3}} + 3b^{\frac{1}{3}}c^{\frac{2}{3}}}|^{\frac{1}{3}}$ #

Alſo $b^{\frac{1}{2}} - c^{\frac{1}{2}}$ is $= \overline{b + c - 2\overline{bc}|^{\frac{1}{2}}}|^{\frac{1}{2}}$.

The Arithmetick of Compound and Univerſal Surds, depends upon the Rules above given about Simple Surds.

* because the Square of $b^{\frac{1}{2}}+c^{\frac{1}{2}}$ or $\sqrt{b}+\sqrt{c}$ is $b+2\sqrt{bc}+c$, or because the Squares of $b^{\frac{1}{2}}+c^{\frac{1}{2}}$ and $\overline{b+c+2\sqrt{bc}}|^{\frac{1}{2}}$ are equal

because the Cubes of $\overline{b+c+3b^{\frac{2}{3}}c^{\frac{1}{3}}+3b^{\frac{1}{3}}c^{\frac{2}{3}}}|^{\frac{1}{3}}$ and $b^{\frac{1}{3}}+c^{\frac{1}{3}}$ are equal.

PART VII.

Concerning the Nature of Æquations, and how to prepare them for a Solution.

AN Æquation is the mutual comparing of two equal Quantities differently expreſt, (which are call'd the two Parts thereof) and is uſually denoted with the Sign = between them, and the ſingle Quantities of both Parts are call'd the Terms of the Æquation;

Thus $a = c + d^3 + eg$ is an Æquation, In which the ſingle Quantity *a*, one Part thereof, is equal to the Compound Quantity $c + d^3 + eg$, the other Part; and the Terms of both Parts, are the ſingle Quantities *a*, *c*, d^3 and *eg*.

An Æquation is two-fold, *viz.* 1. Pure or Simple, or 2. Affected or Compound.

A Pure or Simple Æquation, is that wherein the Quantity ſought (as ſuppoſe *a*) is expreſt by one Power only, as by the 1*ſt*, 2*d*, 3*d*, *&c.* Power; thus $2a + c = d + f$, and $ca^3 = d + b$, are Simple Æquations.

An Affected or Compound Æquation is that in which there are two or more different Degrees or Powers of the Quantity ſought (*a*), As in this Æquation $a^3 + 2a^2 - ca^5 = cd$, there are three different Powers of *a*, *viz.* a^3, a^2 and a^5.

When any Queſtion or Problem is propos'd to be Analytically reſolved; it is very requiſite that the true Deſign, or meaning thereof, be fully and clearly comprehended (in all its Parts) that ſo it may be truly abſtracted from ſuch ambiguous Words, as Queſtions of this kind are often diſguis'd with; otherwiſe it will be very difficult, if not impoſſible, to ſtate the Queſtion Right in its ſubſtituted Letters, and ever to bring it to an Æquation, by ſuch various Methods of ordering thoſe Letters, as the Nature of the Queſtion may require.

Now the Knowledge of this difficult Part of the Work, is only to be obtained by Practice, and a careful minding the Solution of ſuch leading Queſtions, as are in themſelves very eaſy.

Having got ſo clear a Notion of the Queſtion propos'd, as to place down all the Quantities concern'd in their due Order; *viz.*

all the Subſtituted Letters in ſuch Order, as the Nature of the Queſtion requires; the next thing muſt be to conſider whether it be Limited or not, that is, whether it admits of more Anſwers than one; and to diſcover that, Obſerve the following Rules.

Rule 1.

When the Number of Quantities ſought, exceed the Number of the given Æquations, the Queſtion is capable of innumerable Anſwers.

Example.

Suppoſe a Queſtion were propos'd thus; There are three ſuch Numbers, that if the Firſt be added to the Second, their Sum will be 22, and if the Second be added to the Third, their Sum will be 46. What are thoſe Numbers?

Let the three Numbers be repreſented by three Letters thus; call the Firſt a, the Second e, and the Third y.

Then $\left\{\begin{matrix} a+e=22 \\ e+y=46 \end{matrix}\right\}$ according to the Queſtion.

Here the Number of Quantities ſought are Three, *viz.* a, e, and y, and the Number of the given Æquations are but two; therefore this Queſtion is not Limited, but admits of various Anſwers; becauſe for any one of thoſe three Letters, you may take any Number at Pleaſure that is leſs than 22, which with a little Conſideration, will be very eaſy to conceive.

Rule 2.

When the Number of the given Æquations (not depending upon one another) are juſt as many as the Number of the Quantities ſought; then is the Queſtion truly Limited, *viz.* each Quantity ſought hath but one ſingle Value.

As for Inſtance, Let the aforeſaid Queſtion be propos'd thus.

There are three Numbers (a, e, and y, as before) if the Firſt be added to the Second, their Sum will be 22; if the Second be added to the Third, their Sum will be 46; and if the Firſt be added to the Third, their Sum will be 36. What are the Numbers?

That is, $a+e=22$; $e+y=46$; and $a+y=36$.

Now the Queſtion is perfectly Limited, each ſingle Quantity having but one ſingle Value, to wit $a = 6$; $e = 16$; and $y = 30$.

N. B. If the Number of the given Æquations exceeds the Number of Quantities ſought; they not only limit the Queſtion, but oftentimes render it impoſſible, by being propos'd inconſiſtent one to another.

Having truly ſtated the Queſtion in its ſubſtituted Letters, and found it limited to an Anſwer (or at leaſt ſo bounded as to have a certain determinate Number of Anſwers) then let all thoſe ſubſtituted Letters be ſo ordered or compared together, either by Adding, Subſtracting, Multiplying or Dividing them, *&c.* according as the Nature of the Queſtion requires, until all the unknown Quantities except One, are caſt off or vaniſhed; but therein great Care muſt be taken to keep them to an exact Equality; and when that unknown Quantity, or ſome Power of it (as Square, Cube, *&c.*) is found equal to thoſe that are known; then the Queſtion is ſaid to be brought to an Æquation, and conſequently to a Solution, *viz.* fitted for an Anſwer.

But no particular Rules can be preſcribed for the caſting off, or getting away Quantities out of an Æquation; that part of the Art is only to be obtained by Care and Practice. And when that is done, it generally happens ſo, that the unknown Quantity which is retained in the Æquation, is ſo mix'd and entangled with thoſe that are known; that it often requires ſome Trouble and Skill to bring it (or its Powers, *&c.*) to one Side of the Æquation, and thoſe that are known to the other Side; (ſtill keeping them to a juſt Equality) which the ingenious *Van Schooten* in his *Principia Matheſeos Univerſalis* calls Reduction of Æquations.

The Buſineſs of reducing Æquations (as of moſt, if not all Algebraick Operations) is grounded and depends upon a right Application of the five *Axioms* propoſed in Page 6. and therefore, if thoſe *Axioms* be well underſtood, the Reaſon of ſuch Operations muſt needs appear very plain, and the Work be eaſily perform'd; as in the following *Sections.*

Sect. 1. Of Reduction by Addition.

Reduction by Addition is grounded upon *Axiom* 1, and is only the Tranſpoſing (*viz.* the removing) of any Negative Quantity from either Side of an Æquation to the other Side, with the Sign + before it; as in theſe

Examples.

Suppoſe	1	$a - b = d$
Then	2	$a = d + b$
For	3	$b = b$
$1 + 3$	4	$a = d + b$

Again,

Let	1	$a^2 - d = c - a^2$
$1 + d$	2	$a^2 = c - a^2 + d$
$2 + aa$	3	$2a^2 = c + d$

Let	1	$3a - 4 = 6 - a$
$1 + \overline{4}$	2	$3a = 6 + 4 - a$
$2 + a$	3	$4a = 6 + 4 = 10$

Note, *When any abſolute Number is Regiſter'd in the Margin, you draw a Line over it, to diſtinguiſh it from the other Numbers. As* $\overline{4}$ *in the 2d Step of this Example.*

Let	1	$aa - dc - b = dd - 2ba$
$1 + b$	3	$aa - dc = dd - 2ba + b$
$2 + dc$	3	$aa = dd - 2ba + b + dc$
$3 + 2ba$	4	$aa + 2ba = dd + b + dc$

Suppoſe	1	$2da - d = cc - 3baa - aaa$
$1 + aaa$	2	$aaa + 2da - d = cc - 3baa$
$2 + 3baa$	3	$aaa + 3baa + 2da - d = cc$
$3 + d$	4	$aaa + 3baa + 2da = cc + d$

Sect. 2. Reduction by Subſtraction.

Reduction by Subſtraction is grounded upon *Axiom* 2. and perform'd by Tranſpoſing (or removing) any Affirmative Quantity from either ſide of the Æquation, to the other ſide, with the Sign — before it. As in theſe

Examples.

Suppoſe	1	$a + b = d$
And	2	$b = b$
$1 - 2$	3	$a = d - b$

Let	1	$3a + 4 = 6 + a$
$1 - a$	2	$2a + 4 = 6$
$2 - \overline{4}$	3	$2a = 6 - 4 = 2$

Suppoſe	1	$aa + dc + b = dd + 2ba$
$1 - 2ba$	2	$aa - 2ba + dc + b = dd$
$2 - dc$	3	$aa - 2ba + b = dd - dc$
$3 - b$	4	$aa - 2ba = dd - dc - b$

Let	1	$a^3 + d = c^2 + 3ba^2 + 2da$
$1 - 3ba^2 - 2da - d$	2	$a^3 - 3ba^2 - 2da = c^2 - d$

Sect. 3. Of Reduction by Multiplication.

Fractional Quantities, in any Æquation, are brought into whole Quantities, by Multiplying each Term in the Æquation with the Denominators of the Fractions, by *Axiom* 3. As in theſe

Examples.

Suppoſe	1	$\frac{a}{5}=6$
Then	2	$a=6\times 5=30$. For $\frac{a}{5}\times 5=\frac{5a}{5}=a$

Let	1	$3a=\frac{dc}{2b}$	Suppoſe	1	$a=\frac{dd}{a-b}$
$1\times 2b$	2	$6ba=dc$	$1\times\overline{a-b}$	2	$aa-ba=dd$

Suppoſe	1	$\frac{aa}{b}+c+f=\frac{dx}{a}$
$1\times b$	2	$aa+bc+bf=\frac{bdx}{a}$
$2\times a$	3	$a^3+bac+bfa=bdx$

Suppoſe	1	$\frac{aaa}{aa-bb}=\frac{ba-bb}{a+b}$
$1\times\overline{aa-b^2}$	2	$a^3=\frac{ba^3-b^2a^2-b^3a+b^4}{a+b}$
$2\times\overline{a+b}$	3	$a^4+ba^3=ba^3-b^2a^2-b^3a+b^4$

Sect. 4. Of Reduction by Diviſion.

When any Quantity (either known or unknown) is in every Term of an Æquation; if the whole Æquation be Divided by that Quantity, it will be Reduc'd into Lower Terms, per *Axiom* 4. As in theſe

Examples.

Suppoſe	1	$baa+bca=bcd$	Let	1	$aa=7a$
$1\div b$	2	$aa+ca=cd$	$1\div a$	2	$a=7$

Let

Let	1	$ffaa + ffcaa - ffa = ffda + ffdda$
$1 \div ff$	2	$aa + caa - a = da + dda$
$2 \div a$	3	$a + ca - 1 = d + dd$

Or when the unknown Quantity is Multiplied (*viz.* join'd) with any that is known; let the whole Æquation be Divided by the known Quantity, that ſo the unknown may be cleared. As in theſe

Examples.

Suppoſe	1	$ba - ca = d$	Let	1	$ca^2 - da^2 = cd - dd$
$1 \div \overline{b - c}$	2	$a = \frac{d}{b - c}$	$1 \div \overline{c - d}$	2	$a^2 = \frac{cd - dd}{c - d} = d$

Suppoſe	1	$b^2a^3 - 2b^2a^2 = bda + cba$
$1 \div ba$	2	$ba^2 - 2ba = d + c$
$2 \div b$	3	$a^2 - 2a = \frac{d + c}{b}$

Let	1	$49da^2 + 42a^2 = 7bca + 21ca$
$1 \div 7$	2	$7da^2 + 6a^2 = bca + 3ca$
$2 \div a$	3	$7da + 6a = bc + 3c$
$3 \div \overline{7d + 6}$	4	$a = \frac{bc + 3c}{7d + 6}$

Sect. 5. Reduction by Involution.

When there happens to be an Æquation, between any Homogeneal or like Surds; Take away the Indices, or Radical Signs from the Quantities, and they will become Rational. As in theſe

Examples.

Suppoſe	1	$\sqrt{d} = \sqrt{a + c}$	Let	1	$\overline{a^2}\vert^{\frac{1}{3}} = \overline{db + bc}\vert^{\frac{1}{3}}$
1 Ⓖ 2	2	$a = d + c$	1 Ⓖ 3	2	$a^2 = db + bc$

Or if one ſide of the Æquation conſiſts of Surd Quantities, and the other ſide be Rational; then Involve the Rational Quantities to the ſame Power (or Height) with the Number prefix'd to the Radical Sign of the Surd, and then take away the Radical Sign of the ſaid Surd (as in the Four following *Examples*): Or if both

Sides of the given Æquation be Surds of different kinds, Reduce them to the ſame leaſt common Index (by Part 6. Chap. 2. *Prop.* 1.) and then from both Sides take away the ſaid common Index (as in *Example* 5.).

Examples.

$$\begin{array}{c|c|l} \text{Let} & 1 & \sqrt[2]{a} = 6 \\ 1 \odot 2 & 2 & a = 36 \end{array} \qquad \begin{array}{c|c|l} \text{Let} & 1 & \overline{a}|^{\frac{1}{2}} = b + d \\ 1 \odot 3 & 2 & a = b^2 + 2bd + d^2 \end{array}$$

$$\begin{array}{c|c|l} \text{Suppoſe} & 1 & \overline{aa - ba}|^{\frac{1}{3}} = d \\ 1 \odot 3 & 2 & aa - ba = ddd \end{array} \qquad \begin{array}{c|c|l} \text{Let} & 1 & \sqrt[5]{aa} = 7 \\ 1 \odot 5 & 2 & aa = 16807 \end{array}$$

$$\begin{array}{c|c|l} \text{Let} & 1 & \sqrt[2]{a} = \sqrt[3]{b} \\ \left.\begin{array}{c}\text{1 reduc'd to a} \\ \text{common Index.}\end{array}\right\} & 2 & \overline{a^3}|^{\frac{1}{6}} = \overline{b^2}|^{\frac{1}{6}} \\ 2 \odot 6 & 3 & a^3 = b^2 \end{array}$$

Sect. 6. Of Reduction by Evolution.

When any Single Power of the unknown Quantity, is on one Side of an Æquation; Evolve both Sides of the Æquation, according as the Index of that Power denotes, and their Roots will be equal. As in theſe

Examples.

$$\begin{array}{c|c|l} \text{Suppoſe} & 1 & aa = 36 \\ 1 \omega 2 & 2 & a = \overline{36}|^{\frac{1}{2}} = 6 \end{array} \qquad \begin{array}{c|c|l} \text{Let} & 1 & a^3 = 27 \\ 1 \omega 3 & 2 & a = \sqrt[3]{27} = 3 \end{array}$$

$$\begin{array}{c|c|l} \text{Suppoſe} & 1 & aa = lb - dd \\ 1 \omega 3 & 2 & a = \sqrt{bb - dd} \end{array} \qquad \begin{array}{c|c|l} \text{Let} & 1 & a^3 = b^3 + 3b^2c + 3bc + c^3 \\ 1 \omega 3 & 2 & a = b + c \end{array}$$

Or if any Compound Power of the unknown Quantity be at one Side of the Æquation *(that hath a true Root of its kind)* Evolve both the Sides of the Æquation, and it will be depreſs'd to lower Terms. As in theſe

Examples.

$$\begin{array}{c|c|l|l} \text{Suppoſe} & 1 & aa + 2ba + bb = dd & aa - 2ba + bb = ddcc \\ 1 \omega 2 & 2 & a + b = d & a - b = dc. \end{array}$$

Here follow a few Examples of clearing Æquations, wherein all the foregoing Reductions are promiſcuouſly uſed, as occaſion requires.

Example 1.

Suppoſe	1	$\frac{a^2+c-d}{4}=\frac{g-aa}{b}$; What is $a=$ to ?
1×4	2	$a^2+c-d=\frac{4g-4aa}{b}$
$2\times b$	3	$ba^2+bc-bd=4g-4aa$
$3+4a^2$	4	$ba^2+4a^2+bc-bd=4g$
$4+bd$	5	$ba^2+4a^2+bc=4g+bd$
$5-bc$	6	$ba^2+4a^2=4g+bd-bc$
$6\div\overline{b+4}$	7	$a^2=\frac{4g+bd-bc}{b+4}$
7 ɯ 2	8	$a=\left.\overline{\frac{4g+bd-bc}{b+4}}\right\vert^{\frac{1}{2}}$. As was required.

Example 2.

Suppoſe	1	$\frac{a+354}{a}=\frac{3a}{354-a}$; What is the Value of a?
$1\times a$	2	$a+354=\frac{3aa}{354-a}$
$2\times\overline{354-a}$	3	$125316-aa=3aa$
$3+aa$	4	$125316=4aa$
$4\div 4$	5	$31329=aa$
5 ɯ 2	6	$\sqrt{31329}=a=177$ the Value of a required.

Example 3.

Suppoſe	1	$\left.\overline{\frac{a^2+3b^2}{4}}\right\vert^{\frac{1}{2}}-\left.\overline{\frac{a^2-3b^2}{4}}\right\vert^{\frac{1}{2}}=\left.\overline{\frac{baa}{c}}\right\vert^{\frac{1}{2}}$. $a=$?
1 ⊖ 2	2	$\left\{\begin{array}{l}\frac{aa+3bb}{4}-2\times\left.\overline{\frac{aa+3bb}{4}}\right\vert^{\frac{1}{2}}\times\left.\overline{\frac{aa-3bb}{4}}\right\vert^{\frac{1}{2}}\\+\frac{aa-3bb}{4}=\frac{baa}{c}\end{array}\right.$

That is	3	$\frac{aa}{2} - \overline{\frac{a^4 - 9b^4}{4}}\Big\|^{\frac{1}{2}} = \frac{baa}{c}$
For		$\frac{aa + 3bb}{4} + \frac{aa - 3bb}{4} = \frac{2aa}{4} = \frac{a^2}{2}$
And		$2 \times \overline{\frac{aa + 3bb}{4}}\Big\|^{\frac{1}{2}} = \sqrt{aa + 3bb}$
Then		$\sqrt{aa + 3bb} \times \overline{\frac{aa - 3bb}{4}}\Big\|^{\frac{1}{2}} = \overline{\frac{a^4 - 9b^4}{4}}\Big\|^{\frac{1}{2}}$
$3 + \sqrt{\frac{a^4 - 9b^4}{4}}$	4	$\frac{aa}{2} = \frac{baa}{c} + \overline{\frac{a^4 - 9b^4}{4}}\Big\|^{\frac{1}{2}}$
$4 - \frac{ba^2}{c}$	5	$\frac{aa}{2} - \frac{baa}{c} = \overline{\frac{aaaa - 9bbbb}{4}}\Big\|^{\frac{1}{2}}$
5 ◎ 2	6	$\frac{a^4}{4} - \frac{ba^4}{c} + \frac{b^2a^4}{cc} = \frac{a^4 - 9b^4}{4}$
$6 + \frac{ba^4}{c}$	7	$\frac{a^4}{4} + \frac{b^2a^4}{cc} = \frac{a^4 - 9b^4}{4} + \frac{ba^4}{c}$
$7 - \frac{a^4 - 9b^4}{4}$	8	$\frac{b^2a^4}{cc} + \frac{9b^4}{4} = \frac{baaaa}{c}$
$8 \div b$	9	$\frac{ba^4}{cc} + \frac{9b^3}{4} = \frac{a^4}{c}$
$9 \times cc$	10	$ba^4 + \frac{9bbbcc}{4} = ca^4$
10×4	11	$4ba^4 + 9b^3c^2 = 4ca^4$
$11 - 4ba^4$	12	$9b^3c^2 = 4ca^4 - 4ba^4$
$12 \div \overline{4c - 4b}$	13	$a^4 = \frac{9b^3c^2}{4c - 4b}$
For		$\overline{4c - 4b} \times a^4 = 4ca^4 - 4ba^4$
13 ɯ 4	14	$a = \overline{\frac{9bbbcc}{4c - 4b}}\Big\|^{\frac{1}{4}}$. As was required.

By the help of theſe Reductions (properly applyed) the unknown Quantity (*a*) or its Powers are cleared and brought to one Side of an Æquation; and if the unknown Quantity (*a*) is found = to thoſe that are known, then the Queſtion is Anſwer'd. As in the 1*ſt* Examples of *Sect.* 1. and 2.

Or if any ſingle Power of the unknown Quantity (a) is found equal to thoſe that are known, then the reſpective Root of the known Quantities is the Anſwer. As in the firſt Four Examples of *Sect.* 6.

But when the Powers of the unknown Quantity are either mix'd with their Root, as $aa + ba = dd$, *&c.* Or do conſiſt of different Powers; as $a^3 + da^2 = cc$, *&c.* Then they are call'd adfected Æquations; the manner of reſolving which, that is, of finding the Value of a, has been ſhewed in Part IV. Chap. III. And other Methods for the ſame purpoſe ſhall be ſhewed further on.

PART VIII.

Of Proportional Quantities both Arithmetical, Geometrical, and Muſical.

CHAP. I.

Of Arithmetical Proportion.

Definition.

WHen any Rank or Series of Numbers or Quantities, do either increaſe or decreaſe, by an equal Interval or common Difference or Exceſs, they are ſaid to be in Arithmetical Progreſſion continued.

$$\text{As}\left\{\begin{matrix} 2. & 4. & 6. & 8. & 10. & \&c. \\ 15. & 13. & 11. & 9 & 7. & \&c. \end{matrix}\right\}$$

In the former there is a continual Increaſe; in the latter a continual Decreaſe by 2, which is the common Difference or Exceſs.

And univerſally putting a for the 1ſt Term, and e for the common Exceſs or Difference, the Terms will be

$a,\ a+e,\ a+2e,\ a+3e,$ &c. Increaſing.

$a,\ a-e,\ a-2e,\ a-3e,$ &c. Decreaſing.

But the moſt Simple and Natural Progreſſion is that which begins with o, as

$$o,\quad e,\quad 2e,\quad 3e,\quad 4e,\quad 5e,\ \&c.$$

$$o,\ -e,\ -2e,\ -3e,\ -4e,\ -5e,\ \&c.$$

When it begins with any other Term (as a in the former Progreſſions) it is really a Compound of two Progreſſions, one of Equals (a, a, a, a, &c.) and the other of Arithmetical Proportionals (o, e, $2e$, $3e$, $4e$, &c.)

But when the firſt Term exceeds, or is exceeded by, the 2d Term by the ſame Number or Quantity, that the 3d exceeds, or is exceeded by, the 4th Term; but not by the ſame that the 2d exceeds, or is exceeded by, the 3d Term; then that is ſaid to be a Diſcontinued or Disjunct Arithmetical Progreſſion.

$$\text{So}\left\{\begin{matrix} 1, & 3, & 9, & 11 \\ a, & a+e, & a+7e, & a+8e \end{matrix}\right\}\text{are ſaid to be Disjunct or}$$

Diſcontinued Arithmetical Proportions.

Now in order to the finding out how to reſolve Queſtions concerning theſe Proportionals.

$$\text{Let}\begin{cases} a = \text{the leaſt Term.} \\ y = \text{the greateſt Term.} \\ e = \text{the common Exceſs.} \\ n = \text{Number of Terms.} \\ s = \text{Sum of all the Series or Terms.} \end{cases}$$

Theorem 1.

The Sum of the Extreams (*i.e.* of the leaſt and greateſt Terms) is equal to the Sum of any two Means that are equally diſtant from thoſe Extreams (*i. e.* equal to the Sum of the leſs but one, and greateſt but one, or leſs but two, and greateſt but two, or leſs but three, and greateſt but three, &c. Terms) of any Arithmetical Progreſſion, either continued or diſcontinued; and conſequently, if the Number of Terms be odd, the double of the middle Term is equal to the Sum of the two Extreams, or if the Number of Terms be even, the Sum of the two Extreams is equal to the Sum of the two middle Terms.

Demonſtration.

$$\left\{\begin{matrix} \text{Leaſt Term } a \\ +\ \text{Greateſt Term } y \end{matrix}\right\} = a + y.$$

$$\left\{\begin{matrix} \text{Leaſt Term but one } a + e \\ +\ \text{Greateſt Term but one } y - e \end{matrix}\right\} = a + y.$$

$$\left\{\begin{matrix} \text{Leaſt Term but two } a + 2e \\ +\ \text{Greateſt Term but two } y - 2e \end{matrix}\right\} = a + y.$$

$$\left\{\begin{matrix} \text{Leaſt Term but three } a + 3e \\ +\ \text{Greateſt Term but three } y - 3e \end{matrix}\right\} = a + y.$$

$$\text{\&c.} \qquad = \text{\&c.} \quad \text{Q. E. D.}$$

N. B. *What follows relates to* ÷.

Scholium. 1.

Wherefore by proceeding with the firſt Series of theſe Terms, towit a, $a+e$, $a+2e$, $a+3e$, &c. to the n^{th} Term, which is y; alſo with the other Series of Terms, towit y, $y-e$, $y-2e$, $y-3e$, &c. to the n^{th} Term likewiſe, which is a, and then Adding both Series into one Sum, the ſaid Sum will be equal to double the Sum of either Series, and alſo $=$ to n times $a+y$, that is to ſay

$$2s = \overline{a+y} \times n$$

$$\text{And } s = \frac{\overline{a+y}}{2} \times n = \frac{na+ny}{2}$$

Corollaries.

1. $a = \frac{2s}{n} - y = \frac{2s-ny}{n} = \overline{2s-ny} \times \frac{1}{n}$

2. $n = \frac{2s}{a+y} = 2s \times \frac{1}{a+y}$

3. $y = \frac{2s}{n} - a = \frac{2s-na}{n} = \overline{2s-na} \times \frac{1}{n}$

And the Value of s *is found in the next Line before the preceeding Corollaries,*

Scholium 2. and *Theorem* 2.

In the foregoing $\left\{\begin{array}{l}\text{Increaſing}\\ \text{Decreaſing}\end{array}\right\}$ Series, *viz.*

$\left\{\begin{array}{l}a,\ a+e,\ a+2e,\ a+3e,\ \&c.\\ y,\ y-e,\ y-2e,\ y-3e,\ \&c.\end{array}\right\}$ it is eaſy to perceive that the common Difference e is ſo often $\left\{\begin{array}{l}\text{Added}\\ \text{Subſtracted}\end{array}\right\}$ in the laſt Term of the Series, as are the Number of Terms except the firſt; that is the firſt Term $\left\{\begin{array}{l}a\\ y\end{array}\right\}$ hath no difference $\left\{\begin{array}{l}\text{Added}\\ \text{Subſtracted}\end{array}\right\}$ in it; but the laſt hath ſo many times e $\left\{\begin{array}{l}\text{Added}\\ \text{Subſtracted}\end{array}\right\}$ in it, as are the Number of Terms from the firſt to the laſt Incluſive, conſequently the difference of the Extreams is $= e \times$ the Number of all the Terms leſs Unity or 1; that is

$$\overline{n-1} \times e = y - a = ne - e$$

$$\text{Wherefore } e = \frac{y-a}{n-1}$$

Corollaries.

1. $a = -ne + e + y.$
2. $n = \frac{e + y - a}{e}$
3. $y = ne - e + a.$

Theorem 3.

Given a, e, n; required y and s.

Solution.

1. $y = a + ne - e = \frac{2s - na}{n}$ by Theorems 2*d* and 1*ſt*.

Then $na + nne - ne = 2s - na$ by Multiplication.

And $2s = 2na + nne - ne$ by Tranſpoſition.

2. Therefore $s = na + \frac{nne - ne}{2}$ by Diviſion.

Theorem 4.

Given a, e, y; required n and s.

Solution.

1. $n = \frac{y - a + e}{e} = \frac{2s}{a + y}$ by Theorems 2*d* and 1*ſt*.

Then $2es = yy - aa + ea + ey$ by Multiplication.

2. Therefore $s = \frac{yy - aa + ea + ey}{2e}$

Theorem 5.

Given a, e, s; required n and y.

Solution.

Since $y = \frac{2s - na}{n} = a + ne - e$ by Theorems 1*ſt* and 2*d*.

Therefore $nne + 2na - ne = 2s$ by Multiplication and Tranſpoſition.

And $nn + \frac{\overline{2a - e}}{e} \times n = \frac{2s}{e}$ by Dividing by e.

By * Compleating the Square

$$nn + \frac{2a - e}{e} \times n + \frac{4aa - 4ae + ee}{4ee} = \frac{2s}{e} + \frac{4aa - 4ae + ee}{4ee}.$$

By Evolution

$$n + \frac{a}{e} - \frac{1}{2} = \overline{\frac{2s}{e} + \frac{4a^2 - 4ae + e^2}{4ee}}\Big|^{\frac{1}{2}} = \frac{\sqrt{2se + aa - ae + \frac{1}{4}ee}}{ee}$$

And by Tranſpoſition

$$1.\ n = \frac{\sqrt{2se + aa - ae + \frac{1}{4}ee}}{ee} - \frac{a}{e} + \frac{1}{2}$$

And Secondly becauſe $n = \frac{2s}{a + y} = \frac{y - a + e}{e}$ by Theorems 1ſt and 2d.

Therefore $yy + ye = 2se - ae + aa$ by Multiplication and Tranſpoſition.

Compleating □: $yy + ye + \frac{1}{4}ee = 2se - ae + aa + \frac{1}{4}ee$:

By Evolution $y + \frac{1}{2}e = \sqrt{2se - ae + aa + \frac{1}{4}ee}$.

2. $y = \sqrt{2se - ae + aa + \frac{1}{4}ee} - \frac{1}{2}e$ by Tranſpoſition.

Theorem 6.

Given a, y, s; required n and e.

Solution.

1. $n = \frac{2s}{y + a} = \frac{y - a + e}{e}$ by Theorems 1ſt and 2d.

Then $2es - ye - ae = yy - aa$ by Multiplication and Tranſpoſition.

2. Therefore $e = \frac{yy - aa}{2s - y - a}$ by Diviſion.

Theorem 7.

Given a, n, s; required y and e.

Solution.

1. $y = \frac{2s - na}{n} = a + ne - e$ by Theorems 1ſt and 2d.

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* See Part 10.

Then $nne - ne = 2s - 2na$ by Multiplication and Tranſpoſition.

2. Therefore $e = \frac{2s - 2na}{nn - n}$ by Diviſion.

Theorem 8.

Given e, n, y; required a and s.

Solution.

Firſt $a = y - ne + e = \frac{2s - ny}{n}$ by Theorems 2*d* and 1*ſt*.

Then $2s = 2ny - nne + ne$ by Multiplication and Tranſpoſition.

2. Therefore $s = \frac{2ny - nne + ne}{2}$ by Diviſion.

Theorem 9.

Given e, n, s; required a and y.

Solution.

Since $y = a + ne - e = \frac{2s - na}{n}$ by Theorems 2*d* and 1*ſt*.

Then $2na = 2s - nne + ne$ by Multiplication and Tranſpoſition.

1. Therefore $a = \frac{2s - n^2e + ne}{2n}$ by Diviſion.

And ſince $a = y - ne + e = \frac{2s - ny}{n}$ by Theorems 2*d* and 1*ſt*.

Then $2ny = 2s + nne - ne$ by Multiplication and Tranſpoſition.

2. Therefore $y = \frac{2s + n^2e - ne}{2n}$ by Diviſion.

Theorem 10.

Given e, y, s; required a and n.

Solution.

Since $n = \frac{2s}{a + y} = \frac{y - a + e}{e}$ by Theorems 1*ſt* and 2*d*.

Then $aa - ae = yy + ye - 2es$ by Multiplication and Tranſpoſition.

By Compleating □ : $aa - ae + \frac{1}{4}ee = yy + ye - 2es + \frac{1}{4}ee$.

By Evolution $a - \frac{1}{2}e = + \sqrt{yy + ye - 2es + \frac{1}{4}ee}$.

1. Therefore by Tranſpoſition.

$$a = \pm \overline{yy + ye - 2es + \frac{1}{4}ee}\big|^{\frac{1}{2}} + \frac{1}{2}e.$$

And becauſe $a = y - ne + e = \frac{2s - ny}{n}$ by Theorems 2*d* and 1*ſt*,

Therefore $nne - 2ny - ne = -2s$ by Multiplication and Tranſpoſition.

And $nn - \frac{\overline{2y + e}}{e} \times n = -\frac{2s}{e}$ by Diviſion.

By Compleating □ : $nn - \frac{\overline{2y + e}}{e} \times n + \frac{4yy + 4ey + ee}{4ee} = -\frac{2s}{e} + \frac{4yy + 4ey + ee}{4ee}$.

By Evolution $n - \frac{\overline{2y + e}}{2e} = \pm \overline{\frac{-2s}{e} + \frac{4yy + 4ey + e^2}{4ee}}\Big|^{\frac{1}{2}} = \pm \frac{\sqrt{-2se + yy + ey + \frac{1}{4}ee}}{e}$

2. By Tranſpoſition $n = \pm \frac{\sqrt{-2se + yy + ey + \frac{1}{4}ee}}{e} + \frac{y}{e} + \frac{1}{2}$.

Theorem 11.

Given n, y, s; required a and e.

Solution.

1. $a = \frac{2s - ny}{n} = y - ne + e$ by Theorems 1*ſt* and 2*d*.

Then $2ny - 2s = nne - ne$ by Multiplication and Tranſpoſition,

2. Therefore $\frac{2ny - 2s}{nn - n} = e$ by Diviſion.

CHAP.

CHAP. II.

Of Geometrical Proportion.

Definition.

WHen a Rank or Series of Numbers or Quantities, do either increaſe by one common Multiplicator, or decreaſe by one common Diviſor, thoſe Numbers or Quantities are ſaid to be in Geometrical Proportion continued.

As $\begin{cases} 2.\ 4.\ 8.\ 16.\ 32.\ \&c. & \text{Here 2 is the common Multiplyer.} \\ 729.\ 243.\ 81.\ 27.\ 9.\ \&c. & \text{Here 3 is the common Diviſor.} \end{cases}$

Alſo $\begin{cases} 1.\ r.\ r^2.\ r^3.\ r^4.\ \&c. & \text{Here } r \text{ is the common Multiplyer.} \\ 1.\ \frac{1}{r}.\ \frac{1}{r^2}.\ \frac{1}{r^3}.\ \frac{1}{r^4}.\ \&c. & \text{Here } r \text{ is the common Diviſor.} \end{cases}$

And Univerſally $\begin{cases} a.\ ar.\ ar^2.\ ar^3.\ ar^4.\ \&c. & \text{Here } r \text{ is the common Multiplyer.} \\ a.\ \frac{a}{r}.\ \frac{a}{r^2}.\ \frac{a}{r^3}.\ \frac{a}{r^4}.\ \&c. & \text{Here } r \text{ is the common Diviſor.} \end{cases}$

Note, the common Multiplyer, or Diviſor is call'd the **Ratio**, and it ſhews the Habitude or Relation the Numbers or Quantities have to one another; *viz.* whether they are Double, Triple, Quadruple, &c. which *Euclid* thus defines.

Ratio (or Rate) is the mutual Habitude or Reſpect of two Magnitudes (and conſequently of two Multitudes) of the ſame kind, each to other, according to Quantity. *Eucl.* 5. *Defin.* 4.

So that there cannot be leſs than three Terms to form a Proportionality, or Similitude of Ratios.

Sect. 1. Of Proportions, Geometrical continued.

In order to ſolve ſuch Queſtions as relate to Geometrical Proportion continued. I will

Suppoſe, $\begin{cases} a = \text{Leaſt Term of any Series.} \\ y = \text{Greateſt Term of the ſaid Series.} \\ s = \text{Sum of all the Terms.} \\ n = \text{Number of Terms.} \\ r = \text{Common Ratio; but Note that } r \text{ muſt be } \sqsubset 1. \end{cases}$

Lemma

Lemma 1.

In any Series in Geometrical Proportion continued, the Rectangle (or Product) of the Extreams is equal to the Rectangle of any two Means that are equally diſtant from thoſe Extreams.

Demonſtration.

a being (as we have already ſuppos'd) = leaſt Term of any Series in ∺ and r = common Ratio; the Leaſt, Leaſt but one, Leaſt but two, Leaſt but three, &c. Terms will be a, ar, ar^2, ar^3, &c. And y being alſo = greateſt Term of the ſaid Series; the Greateſt, Greateſt but one, Greateſt but two, Greateſt but three &c. Terms will be y, $\frac{y}{r}$, $\frac{y}{r^2}$, $\frac{y}{r^3}$, &c.

And the Means or middle Terms, we will denote by &c. By which Means our ſaid Series in ∺ $\left\{\begin{matrix}\text{Increaſing}\\ \text{Decreaſing}\end{matrix}\right\}$ may be writ

$$\text{Thus}\left\{\begin{matrix} a,\ ar,\ ar^2,\ ar^3,\ \&c.\ \frac{y}{r^3},\ \frac{y}{r^2},\ \frac{y}{r},\ y. \\ y,\ \frac{y}{r},\ \frac{y}{r^2},\ \frac{y}{r^3},\ \&c.\ ar^3,\ ar^2,\ ar,\ a. \end{matrix}\right.$$

And $a \times y$ is evidently $= ar \times \frac{y}{r} = ar^2 \times \frac{y}{r^2} = ar^3 \times \frac{y}{r^3}$, &c. Q. E. D.

Lemma 2.

If from the Sum of any Series in ∺ the Leaſt and greateſt Terms be ſeverally Subſtracted; I ſay, that the firſt Remainder is equal to the Product of the ſecond Remainder, and of the common Ratio.

Demonſtration.

$$a + ar + ar^2 + ar^3 + \&c. + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r} + y$$
$$= y + \frac{y}{r} + \frac{y}{r^2} + \frac{y}{r^3} + \&c. + ar^3 + ar^2 + ar + a$$
$$= s.$$

Wherefore by Tranſpoſition

$$ar + ar^2 + ar^3 + \&c. + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r} + y = s - a.$$

And $a + ar + ar^2 + ar^3 + \&c. + \frac{y}{r^3} + \frac{y}{r^2} + \frac{y}{r} =$ $s - y.$

But

Suppoſe a Series forwards and backwards to
$2a+4a+8a+16a = s = 16a+8a+4a+2a+a;$

But the firſt of the two next foregoing Series is manifeſtly equal to the ſecond $\times r$.

That is to ſay $\overline{s-a} = \overline{s-y} \times r$.

$$\text{Wherefore}\begin{cases} s = \dfrac{yr-a}{r-1} \\ y = \dfrac{sr-s+a}{r} \\ a = s - sr + yr \\ r = \dfrac{s-a}{s-y} \end{cases}$$

1 [illegible]

2 6 [illegible]

Scholium.

It is manifeſt by viewing the foregoing Series in ∺ that $ar^{n-1} = y$; for the Exponent of r in the leaſt Term of the ſaid Series is 0, in the leaſt Term but one is 1, in the leaſt Term but two is 2; and therefore Univerſally the Exponent of r in the n^{th} or greateſt Term of the ſaid Series is $n-1$; wherefore $ar^{n-1} = y$.

And by Dividing each Part by r^{n-1}.

$$a = \frac{y}{r^{n-1}}$$

And by Dividing each Part of the 1*ſt* Step by a, and then Extracting out of each the $\overline{n-1}$ Root

$$r = \overline{\frac{y}{a}}\Big|^{\frac{1}{n-1}}$$

Or by Multiplying each Part of the 1*ſt* Step by $\frac{r}{a}$ we have

$$r^n = \frac{yr}{a}$$

* Now the Logarithm of r^n is (as ſhall be ſhew'd when we come to treat of Logarithms; but in the mean time we will ſuppoſe L to ſtand for the Word Logarithm) $= n \times \text{L}, r$.

And the Logarithm of $\frac{yr}{a}$ is $= \text{L}, y + \text{L}, r - \text{L}, a$; wherefore (ſince the Logarithms of equal Quantities are equal) $\text{L}, y + \text{L}, r - \text{L}, a = n \times \text{L}, r$.

Conſequently $\dfrac{\text{L}, y + \text{L}, r - \text{L}, a}{\text{L}, r} = n$.

is $\underline{2a+4a+8a+16a} = 30a = s-a$,

and $\overline{8a+4a+2a+a} \times 2 = 30a = \overline{s-16a} \times 2$.

Here 2 is the common Ratio.

* Or $\frac{yr}{a}$ being $= r^n$

$\frac{yr}{a} \div) r = n$; that is to ſay, $\frac{yr}{a}$ being continually Divided by r, until ſome Quotient be $= 1$ (or $\sqsupset 1$) the Number of thoſe Diviſions, is $= n$ (or $\sqsubset n$).

Propoſitions.

I. Given a, r, n; required y, s.

Solution.

1. $y = ar^{n-1}$ by the foregoing *Scholium.*

And $s = \frac{yr - a}{r - 1}$ by *Lemma* 2.

Therefore $sy = \frac{ayr^n - a^2r^{n-1}}{r - 1} = \frac{ayr^n - ay}{r - 1}$ (becauſe $ar^{n-1} = y$).

2. And conſequently $s = \frac{ar^n - a}{r - 1}$ by Dividing each Part by y.

II. Given a, r, y; required s, n.

Solution.

1. $s = \frac{yr - a}{r - 1}$ by *Lemma* 2.

2. $n = \frac{L, y - L, a}{L, r} + 1$ by the laſt *Scholium.*

III. Given a, r, s; required y, n:

Solution.

1. $y = \frac{sr - s + a}{r} = ar^{n-1}$ by *Lemma* 2. and its *Scholium*:

$\frac{sr - s + a}{a} = r^n$ by Multiplying each Part by $\frac{r}{a}$

But $n \times L, r = L, \overline{\frac{sr - s + a}{a}}$ (by the Nature of Logarithms).

$$2.\left\{\begin{array}{l}\text{Therefore } n = L, \frac{sr - s + a}{a} \div L, r = \\ \frac{L, \overline{sr - s + a} - L, a}{L, r}\end{array}\right.$$

IV. Given a, y, s; required r, n.

Solution.

1. $r = \frac{s-a}{s-y}$ by *Lemma* 2.

2. And $n = \frac{L,\, y - L,\, a}{L,\, r} + 1$ (by the foregoing *Scholium*) $= \frac{L,\, y - L.\, a}{L,\, s-a - L,\, s-y} + 1$ (becauſe $r = \frac{s-a}{s-y}$).

V. Given a, n, s; required r, y.

Solution.

Since $\frac{sr - s + a}{r} = y = ar^{n-1}$ by *Lemma* 2, and its *Scholium.*

1. Therefore $-r^n + \frac{s}{a}\, r = \frac{s-a}{a}$ by Multiplication, Transpoſition and Diviſion.

And ſince $y = ar^{n-1}$ and $r = \frac{s-a}{s-y}$; and so $ar^{n-1} = a \times \overline{\frac{s-a}{s-y}}\Big|^{n-1}$ *

2. Therefore $y \times \overline{s-y}|^{n-1} = a \times \overline{s-a}|^{n-1}$.

$* = a \times \frac{\overline{s-a}|^{n-1}}{\overline{s-y}|^{n-1}} = y$

VI. Given a, n, y; required r, s.

Solution.

1. $r = \overline{\frac{y}{a}}\Big|^{\frac{1}{n-1}}$ by the foregoing *Scholium.*

And $\frac{yr - a}{r - 1} = s$ (by *Lemma* 2.)

2. Therefore $s = \frac{y \times \overline{\frac{y}{a}}\Big|^{\frac{1}{n-1}} - a}{\overline{\frac{y}{a}}\Big|^{\frac{1}{n-1}} - 1}$ (by Subſtitution) $= \frac{y-a}{\overline{\frac{y}{a}}\Big|^{\frac{1}{n-1}} - 1} + y$. For, if we substitute x for $\overline{\frac{y}{a}}\Big|^{\frac{1}{n-1}}$, then

$s = \frac{y \times \overline{\frac{y}{a}}\Big|^{\frac{1}{n-1}} - a}{\overline{\frac{y}{a}}\Big|^{\frac{1}{n-1}} - 1}$ is $= \frac{yx - a}{x - 1} = \frac{y-a}{x-1} + \frac{yx-y}{x-1}$ VII.

$-\, y-a$ . . . is $= \underline{yx-y}$.

VII. Given r, n, y; required a, s.

Solution.

1. $a = \frac{y}{r^{n-1}}$ (by the next foregoing *Scholium.*)

But $\frac{yr - a}{r - 1} = s$ (by *Lemma* 2.).

2. Therefore $s = \frac{yr - \frac{y}{r^{n-1}}}{r - 1} = \frac{yr^n - y}{r^n - r^{n-1}}$ (by Subſtitution).

VIII. Given r, n, s; required a, y.

Solution.

Since $\frac{sr - s + a}{r} = y = ar^{n-1}$ (by *Lemma* 2, and its *Scholium.*)

1. Therefore $\frac{sr - s}{r^n - 1} = a$ (by Multiplication, Tranſpoſition, and Diviſion.)

And ſince $s = \frac{yr^n - y}{r^n - r^{n-1}}$, by *Propoſition* VII.

2. Therefore $\frac{sr^n - sr^{n-1}}{r^n - 1} = y$ (Multiplying each Part by $\frac{r^n - r^{n-1}}{r^n - 1}$)

IX. Given r, y, s; required a. n.

Solution.

1. $a = s - sr + yr$ (by *Lemma* 2.)

But $\frac{y}{a} \left(= \frac{y}{s - sr + yr}\right) = r^{n-1}$ (by the next foregoing *Scholium.*)

And $\mathrm{L},\ \frac{y}{s - sr + yr} = \overline{n - 1} \times \mathrm{L},\ r$ (by the Nature of Logarithms) $= n \times \mathrm{L},\ r - \mathrm{L},\ r$.

2. Therefore $\mathrm{L},\ \frac{y}{s - sr + yr} + \mathrm{L},\ r \div \mathrm{L},\ r = n$ (by Tranſpoſition and Diviſion.)

That is $n = \frac{L,\ y - L,\ \overline{s - sr + yr} + L,\ r}{L,\ r}$

X. Given n, y, s; required r, a.

Solution.

Since, by *Propoſition* VIII, $sr^n - sr^{n-1} = yr^n - y$:

1. Then $-r^n + \frac{sr^{n-1}}{s-y} = \frac{y}{s-y}$ (by Tranſpoſition and Diviſion.)

2. And (by *Propoſition* V.) $a \times \overline{s-a}|^{n-1} = y \times \overline{s-y}|^{n-1}$.

Scholium.

If a be = finite Number, and n = *an* infinite (or infinitely great) Number; then (r being = finite Number, and by *Suppoſition* $\sqsubset 1$.) y (by *Propoſition* I.) $= ar^{n-1}$, muſt be = an infinite Number; and conſequently s (by *Propoſition* I) $= \frac{ar^n - a}{r-1}$ muſt be = an infinite Number;

But if y be = a finite Number, and n = an infinite one, then (r being finite and $\sqsubset 1$.) a (by *Propoſition* VII.) $= \frac{y}{r^{n-1}}$ will be = an infinitely little Number, and the only way we have to write ſuch a Number, is by o.

Wherefore $s = \frac{yr^n - y}{r^n - r^{n-1}}$ (by *Propoſition* VII.) $= \frac{yr - a}{r - 1}$ will be, in this Caſe, ^(because a is infinitely little in resp of 1 $= \frac{yr}{r-1}$; that is = a finite Number.

Upon this *Canon* are founded ſuch Queſtions as are uſually propos'd in infinite Decreaſing Geometrical Proportion, which are eaſily ſolv'd by the ſaid *Canon*; to wit $\frac{yr}{r-1} = s$. As for Inſtance;

If it were required to find the Sum of this Decreaſing Geometrical Proportion $\frac{AB}{bc}, \frac{AB}{b^2c^2}, \frac{AB}{b^3c^3}$, &c. *In infinitum.*

Here $\frac{AB}{bc} = y$.
and $bc = r$.

☞ *Where Note that* $\frac{AB}{bc}$ *muſt be = finite Number*, and $bc \sqsubset 1$.

Whence $\frac{yr}{r-1} = \frac{AB}{bc} \times bc \div \overline{bc - 1} = \frac{AB}{bc - 1} = s$.

And by Subſtituting o for *z*, in each of the foregoing X *Propoſitions*, you'll have *Canons* for any Propoſition in infinite Decreaſing Geometrical Proportion continued. As for Inſtance;

a, *y*, *s*, being given in order to find *r* in an infinite Decreaſing Series in ∺.

I find in *Propoſition* IV. that in a finite Series in ∺, $r = \frac{s - a}{s - y}$; wherefore in an infinite Decreaſing one $r = \frac{s}{s - y}$. *Et ſic de Cæteris.*

Sect. 2. Of Proportions Disjunct.

When the firſt Term has the ſame Ratio to the ſecond, that the third hath to the fourth Term; but not the ſame Ratio which the ſecond hath to the third Term: that Proportion is ſaid to be Diſcontinued or Disjunct.

So $\left\{\begin{matrix} 2, & 4, & 18, & 36, \\ 36, & 18, & 4, & 2, \end{matrix}\right\}$ or Univerſally *a*, *ae*, *d*, *de*,

Are ſaid to be in Geometrical Proportion Disjunct; for the Ratio which 2 hath to 4, (which is 2); 18 hath to 36; but 4 hath not to 18: and the ſame Ratio which *a* hath to *ae* (which is *e*) *d* hath to *de*, but *ae* hath not to *d*.

Theorem.

In any Geometrical Proportion Disjunct, the Product of the Means is equal to the Product of the Extreams; that is to ſay, ſince

$$a \cdot\cdot ae :: d \cdot\cdot de$$

I ſay $a \times de = ae \times d$

Demonſtration.

It is evident that $ade = aed$. *Q. E. D.*

☞ *Note that* Euclid *in his 5th Book expreſſes the Ratio of Proportionals in this manner,* viz. *the Ratio of a to b, thus* $\frac{a}{b}$.

If 4 Quantities are Proportional, they will alſo be Proportional in Alternation, Inverſion, Compoſition, Diviſion, Converſion and Mixtly. *Eucl.* 5. *Def.* 12, 13, 14, 15, 16.

That is, if | 1 | $a \,..\, b :: c \,..\, d$ be in direct Proportion.
Then | 2 | $a \,..\, c :: b \,..\, d$ Alternate. For $ad = bc$.

And 3 $b..a::d..c$ Inverted. For $ad = bc$.
Alſo 4 $a + b..b::c + d..d$ Compounded.
For 5 $da + db = bc + bd$; that is, $ad = bc$, as before.
Or 6 $a + c..c::b + d..d$. Alternately Compounded.
For 7 $ad + cd = cb + cd$; that is, $ad = cb$.
Again 8 $a - b..b::c - d..d$ Divided.
For 9 $ad - bd = bc - bd$; that is, $ad = bc$.
Or 10 $a - c..c::b - d..d$ Alternately Divided.
For 11 $ad - cd = cb - cd$, that is, $ad = bc$.
And 12 $a..b + a::c..d \pm c$ Converted.
For 13 $ad \pm ac = bc + ac$ that is, $ad = bc$.
Laſtly 14 $a + b..a - b::c + d..c - d$ Mixtly.
For 15 $ac - ad + bc - bd = ac + ad - bc - bd$.
That is 16 $2bc = 2ad$ Conſequently $bc = ad$. As at firſt.

Sect. 3. How to turn Æquations into Analogies.

From the next foregoing *Section*, it will be eaſy to conceive how to turn or Diſſolve Æquations into Analogies or Proportion.

For if the Rectangle of two (or more) Quantities, be equal to the Rectangle of two (or more) Quantities; then are thoſe 4 (or more) Quantities Proportional. By the 16. 6. *Euclid.*

That is, if 1 $ab = dc$.
Then 2 $a..c::d..b$.
Or 3 $c..a::b..d$. &c.

From whence there ariſes this general Rule for turning Æquations into Analogies.

Rule.

Divide either Side of the given Æquation (if it can be done) into two ſuch Parts or Factors, as being Multiplyed together, will produce that Side again, and make thoſe two Parts the two Extreams, then Divide the other Side of the Æquation (if it can be done) in the ſame manner as the firſt was, and let thoſe two Parts or Factors be the two Means.

For Inſtance, ſuppoſe $ab + ad = bd$.

Then $a..b::d..b + d$, or $b..a::b + d..d$, &c.

Or, taking ad from both Sides of the Æquation, it will be, $ab = bd - ad$.

Then $a\,..\,d :: b - a\,..\,b$, or $b\,..\,d :: b - a\,..\,a$. &c.

Again ſuppoſe $aa + 2ae = 2by + yy$.

Here a and $\overline{a + 2e}$ are the two Factors of the firſt Side in this Æquation; for $\overline{a + 2e} \times a = aa + 2ae$.

Again y and $2b + y$, are the two Factors of the other Side.

Therefore $a\,..\,y :: 2b + y\,..\,a + 2e$.

Or $2b + y\,..\,a + 2e :: a\,..\,y$. &c.

When one Side of any Æquation can be Divided into two Factors as before, and the other Side cannot be ſo Divided; then make the Square-Root of that Side either the two Extreams, or the two Means.

For inſtance, Suppoſe $\underline{bc + bd} = da + g$.

Then $b\,..\,\sqrt{da + g} :: \sqrt{da + g}\,..\,c + d$.

Or $\sqrt{da + g}\,..\,b :: c + d\,..\,\sqrt{da + g}$. &c.

CHAP. III.

Of Harmonical Proportion.

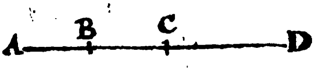
Definition.

MUſical, or Harmonical Proportion, is when of three Quantities (or rather Numbers) the firſt hath the ſame Ratio to the third, as the difference between the firſt and ſecond hath to the Difference between the ſecond and third. As in theſe following.

Suppoſe a, b, c, in Muſical Proportion.

Then	1	$a\,..\,c :: b - a\,..\,c - b$
Therefore	2	$cb - ca = ac - ab$
$2 + ca$	3	$cb = 2ac - ab$
$3 \div \overline{2c - b}$	4	$\frac{cb}{2c - b} = a$ the firſt Term.
$3 + ba$	5	$cb + ba = 2ac$
$5 \div \overline{c + a}$	6	$b = \frac{2ac}{c + a}$ the ſecond Term.
$5 - cb$	7	$ba = 2ac - cb$
$7 \div \overline{2a - b}$	8	$\frac{ba}{2a - b} = c$, the third Term.

A Line AD is said to be divided harmonically in A, B, C, D; when the whole AD has the same proportion to either extreme part that the other extreme AB has to the middle BC;



If there are 4 Terms in Muſical Proportion, the Firſt hath the ſame Ratio to the Fourth, as the Difference between the Firſt and Second hath to the Difference between the Third and Fourth.

That is, Let *a*, *b*, *c*, *d*, be the 4 Terms, &c.

Then	1	$a \,..\, d :: b - a \,..\, d - c$
Therefore	2	$db - da = ad - ac$
$2 + da$	3	$db = 2da - ac$
$3 \div \overline{2d - c}$	4	$\frac{db}{2d - c} = a$
$3 \div d$	5	$b = \frac{2da - ac}{d}$
$3 + ca$	6	$db + ca = 2da$
$6 - db$	7	$ca = 2da - db$
$7 \div a$	8	$c = \frac{2da - db}{a}$
$7 \div \overline{2a - b}$	9	$\frac{ca}{2a - b} = d.$

I have seen ax. ax. ax. ax. I say to u.
I have seen a x. a x. x. yet none so x as u.
Too YY u are. too YY u b.
Too YY u are. I C [illegible]

PART

PART IX.

The Derivation and Compoſition of Quadratick, Cubick, &c. Æquations.

CHAP. I.

The Origination of Quadratick Æquations, is thus Deriv'd by Mr. Harriot.

Caſe 1. If $\left\{\begin{array}{l} a = +b \\ a = -c \end{array}\right\}$ then by Tranſpoſition $\left\{\begin{array}{l} a - b = 0 \\ a + c = 0 \end{array}\right\}$ and by Multiplying one by the other, you'll have

$$aa + \text{[illegible]} - ba - bc = 0.$$

That is $aa + \overline{c - b} \times a - bc = 0$.

Caſe 2. If $\left\{\begin{array}{l} a = -b \\ a = +c \end{array}\right\}$ then by Tranſpoſition $\left\{\begin{array}{l} a + b = 0 \\ a - c = 0 \end{array}\right\}$ and by Multiplication

$$aa - ca + ba - bc = 0.$$

That is $aa - \overline{c - b} \times a - bc = 0$.

Caſe 3. If $\left\{\begin{array}{l} a = +b \\ a = +c \end{array}\right\}$ then by Tranſpoſition $\left\{\begin{array}{l} a - b = 0 \\ a - c = 0 \end{array}\right\}$ And by Multiplying them by one another, the Product is

$$aa - \overline{c + b} \times a + bc = 0.$$

Caſe 4. If $\left\{\begin{array}{l} a = -b \\ a = -c \end{array}\right\}$ then by Tranſpoſition $\left\{\begin{array}{l} a + b = 0 \\ a + c = 0 \end{array}\right\}$ and by Multiplying them by one another, you'll have

$$aa + \overline{c + b} \times a + bc = 0.$$

Hence all adfected Quadratick Æquations, wherein the higheſt Term is a^2, or the Square of the Quantity ſought, are reducible to theſe four Forms.

$$1.\ a^2 + \overline{c - b} \times a - bc = 0.$$
$$2.\ a^2 - \overline{c - b} \times a - bc = 0.$$
$$3.\ a^2 - \overline{b + c} \times a + bc = 0.$$
$$4.\ a^2 + \overline{c + b} \times a + bc = 0.$$

And theſe Mr. *Harriot* properly calls Original Æquations, and from them ſhews that every Quadratick Æquation, wherein the higheſt Term is a^2, or the Square of the Quantity ſought, hath juſt two real Roots according to the Dimenſions of the higheſt Power, as being made up by the Multiplication of two Simple Æquations; and theſe Roots may be both Affirmative, (as in Caſe 3*d.*) or both Negative, (as in Caſe 4*th.*) or one Affirmative, and the other Negative (as in the 1*ſt* and 2*d* Caſes); and ſometimes they are equal to each other, and ſometimes not; and the abſolute Quantity given bc, is always the Rectangle of the two Roots b and c (or of the two Values of a); and if bc hath a poſitive Sign, the two Roots have like Signs, but, if a Negative one, unlike; and the Coefficient of a in the 2*d* Term, is always equal to the Sum of both the Roots with contrary Signs, to what they had at first before Transposition x

x That b or c, with its proper Sign, is one true Root of the foregoing Original Æquations, will appear, by Subſtituting b or c with its proper Sign, and its Square, for, and inſtead of a and its Square, in any of the foregoing Original Æquations.

To inſtance what I here ſay, I will make uſe of the firſt Original Æquation, Namely

$$a^2 + \overline{c - b} \times a - bc = 0.$$

Now if you Subſtitute bb for aa, and b for a in the foregoing Æquation, It will become

$$bb + \overline{c - b} \times b - bc = 0.$$

Which is manifeſtly true. For by Abbreviating this Æquation all its Terms are deſtroyed, and it becomes $0 = 0$.

Whence it is certain, that one of the Roots of (or Values of a in) this Æquation

$$aa + \overline{c - b} \times a - bc = 0, \text{ is } b.$$

In the next Place, I ſay $-c$ is the other Root of the ſaid Original Æquation; for by Subſtituting $+cc$ (*i. e.* the Square of

$-c$) for aa, and $-c$ for a in the ſaid Original Æquation; it becomes

$$cc + \overline{c - b} \times -c - bc = 0:$$

That is $cc - cc + bc - bc = 0$,

Or $0 = 0$.

In like manner you may prove b and c, with their proper Signs, to be the two Roots of any of the other three Original Quadratick Æquations.

Now ſuppoſing $c \sqsubset b$, and $c - b = p$

As alſo $c + b = q$

And $bc = h$

The 4 foregoing Æquations will become = to theſe 4 following ones reſpectively.

1. $aa + pa - h = 0$.
2. $aa - pa - h = 0$.
3. $aa - qa + h = 0$.
4. $aa + qa + h = 0$.

In the firſt of which Caſes, you have the Sum of the two Roots ſought (s) = [illegible] p; in the 2d (s) = [illegible] p; in the 3d Caſe (s) = [illegible] q and in the 4th Caſe (s) = [illegible] q: as alſo the Rectangle (r) = $-h$, in the 1ſt and 2d Caſes; but = $+h$ in the 3d and 4th Caſes, given in order to find their Difference (d); and then each of the ſaid Roots.

The manner of finding all which, ſhall be ſhew'd in Part X.

Now foraſmuch as ſome Algebraiſts define adfected Quadratick Æquations in this manner, *viz.*

'When the Quantity ſought is brought to an Equality with 'thoſe that are known, and is at one Side of the Æquation, in 'no more than two different Powers, whoſe Indices are double 'to one another; thoſe Æquations are called adfected Quadra'tick Æquations.

We may ſay by that Definition,

That $a^4 \pm ba^2 = \pm fg$

Or $a^6 \pm da^3 = \pm c$

Or Univerſally $a^n \pm sa^{\frac{n}{2}} = \pm R$ is an adfected Quadratick Æquation.

But theſe Æquations having as many Roots, Real, or Imaginary, as there are Units in the Index of the higheſt Power of the Root ſought; and being not produc'd by the Multiplication of two Lateral Æquations (as the foregoing Original Quadraticks are), can't be ſaid to be Original adfected Quadratick Æquations; but (ſince the manner of ſolving thoſe, is of the ſame Nature with that of ſolving the foregoing ones) may be call'd adfected Quadratick Æquations.

CHAP. II.

Of the Origination of Cubick, &c. Æquations.

MR. *Harriot* ſhews the Original of a Cubick Æquation, to be deriv'd from three Lateral Æquations, reduc'd firſt to the Form of Binomials, or Reſiduals; and then Multiplyed continually into each other; or elſe from one Quadratick Multiplyed by a Lateral. Whence he deduces, that all Cubick Æquations have three Roots, Real or Imaginary, or as many as are the Dimenſions of its higheſt Power, and no more.

Original Cubick Æquations may be thus form'd.

Let the three Roots be $\left\{\begin{matrix} a = b \\ a = c \\ a = d \end{matrix}\right\}$ Then by Tranſpoſition $\left\{\begin{matrix} a - b = 0 \\ a - c = 0 \\ a - d = 0 \end{matrix}\right\}$ And theſe three Reſiduals Multiplyed continually into each other, will produce this Æquation.

$$\begin{matrix} & -b & & +bc & & \\ a^3 & -c & a^2 & +bd & a & -bcd = 0. \\ & -d & & +cd & & \end{matrix}$$

In like manner he ſhews the Derivation of Biquadratick Æquations, to be from four Simple, or Lateral Æquations reduc'd (as above) to the Form of Reſiduals, or Binomials, and then continually Multiplyed into each other: or elſe from a Cubick into a Lateral: or from one Quadratick into another: or from one Quadratick Multiplyed into two Laterals continually. Wherefore he ſaith, every Biquadratick will have four Roots, Real or Imaginary, agreeable to the Dimenſions of its higheſt Power, and no more

Thus, if the former Cubick be Multiplyed by $a + f = 0$, this Biquadratick Æquation will be produc'd, *viz.*

$$a^4 \begin{array}{l} -b \\ -c \\ -d \\ +f \end{array} a^3 \begin{array}{l} +bc \\ +bd \\ +cd \\ -fb \\ -fc \\ -fd \end{array} a^2 \begin{array}{l} -bcd \\ +fbc \\ +fbd \\ +fcd \end{array} a - fbcd = 0.$$

From which Original of theſe Æquations, 'tis plain, that after you diſcover the Value of any one Root, you may depreſs the Equation a Dimenſion lower, by dividing it by ſuch Root reduc'd to the Form of a Reſidual or Binomial as above.

Thus if you find that one Root, or one a of the laſt foregoing Equation, is $= -f$; then Divide the Equation by $a + f = 0$, and it will bring it down to a Cubick; and that Cubick being again Divided by $a - b = 0$, or $a - c = 0$, or $a - d = 0$, will be depreſs'd into an Original Quadratick, &c. And this is ſometimes of good Uſe to diſſolve Compound Equations into their Components, as ſhall be ſhew'd further on.

From this Method of Compoſition of theſe Æquations, 'tis alſo apparent of what Members each of the Coefficients are made up.

I. For the Coefficient of the 2*d* Term is always the Aggregat or Sum of all the Roots, with contrary Signs. Thus in the Cubick Equation above-mention'd, the Coefficient of the 2*d* Term is $\begin{array}{l} -b \\ -c \\ -d \end{array}$ And $\begin{array}{l} -b \\ -c \\ -d \\ +f \end{array}$ is the Coefficient of the 2*d* Term of the above Biquadratick Equation: And $\begin{array}{l} +b \\ +c \\ +d \end{array}$ in the former, and $\begin{array}{l} +b \\ +c \\ +d \\ -f \end{array}$ in the latter Equation, is the Sum of all the Roots; whence what we have ſaid in the 1*ſt* is manifeſt.

Wherefore it follows, that if all the Negative Roots, ſecluding their Signs, be equal to all the Affirmative ones; (tho' not each to each reſpectively) then will the 2*d* Term quite vaniſh out of the Æquation, and be wanting, becauſe the Affirmatives and Negatives do mutually deſtroy each other. And *vice verſa*, when ever the 2*d* Term is wanting in any of theſe Equations, the Roots are thus equal, and have contrary Signs.

II. The Coefficient of the 3*d* Term is the Aggregat of all the Rectangles made by the Multiplication of every Pair of the Roots (with contrary Signs) as often as they can be taken, which in a Cubick is 3, in a Biquadratick is 6, in a 5th Power is 10. &

&c. according to the order of Triangular Numbers. Thus in the 3*d* Term of the Cubick Equation before mention'd $\begin{matrix}+bc\\+bd\\+cd\end{matrix}$ the Coefficient is the Aggregat of the three Rectangles of the Roots b, c and d, taken by Pairs.

And here, if all the Negative Rectangles (ſecluding their Signs) are equal to all the Affirmative ones, they will deſtroy one another, and ſo the 3*d* Term will vaniſh, or be wanting.

III. The Coefficient of the 4*th* Term is the Aggregat of all the Solids made by the continual Multiplication of all the Ternarys, or every three Roots with contrary Signs, *&c.* and ſo on *ad infinitum.*

IV. As in Quadraticks, the abſolute Number or Quantity given, is always the Rectangle of the two Roots, or Values of a, So in Cubicks, 'tis always the Solid of all the three Roots, with their Signs changed, one into another; and in Biquadraticks of all the four Roots, *&c.*

From this Method of Compoſition of theſe Equations, 'tis alſo Evident, that the Affirmative Roots of any Equation, are changed into Negatives, and the Negative Roots into Affirmatives, by changing the Signs of every other Term of the ſaid Æquation, that is the Signs of the 2*d*, 4*th*, 6*th*, 8*th*, *&c.* Terms, or of the 1*ſt*, 3*d*, 5*th*, 7*th*, *&c.* Thus the Signs of the Roots of this Equation; $a^4 - 4a^3 - 15a^2 + 58a - 40 = 0$, are chang'd by writing it thus, $a^4 + 4a^3 - 15a^2 - 58a - 40 = 0$; Or thus, $-a^4 - 4a^3 + 15a^2 + 58a + 40 = 0$. And the Signs of the Roots of this Equation, $a^3 + pa^2 + qa + r = 0$, are chang'd by writing it thus, $a^3 - pa^2 + qa - r = 0$; Or thus, $-a^3 + pa^2 - qa + r = 0$.

I. *Again,* Since any Original Quadratick Equation may (by what was before ſaid) be Deſign'd by the following one, *viz.*

$$a^2 \begin{matrix}-\overline{+b-p}\\-\overline{+b-q}\end{matrix}\, a + \overline{b^2 \begin{matrix}-p\\-q\end{matrix}\, b} + pq = 0.$$

In which Equation, the two Roots or Values of a are $b - p$, and $b - q$: and ſuppoſe p not $\sqsubset q$, and $\sqsubset 0$; then $b - p$ will be $=$ the greateſt Value of a, which I ſuppoſe is not Imaginary; for if it was, the ſaid Quadratick Equation, when its known Quantities were deſign'd by Figures, would have Imaginary Terms in it, (which I ſuppoſe neither that, nor any of the following ones has) or the other Value of $a = b - q$, muſt be Imaginary too, and of conſequence the Original Quadratick Equation propos'd, have no real Root. Thus if $b - p$ be $= b - y + \surd - z$

$+\sqrt{-z}$, then $b-q$ wou'd be $= b-y-\sqrt{-z}$; that is, if p be $= y-\sqrt{-z}$, then q will be $= y+\sqrt{-z}$, that is, the ſame with the other, only with a contrary Sign to the Imaginary Term, as is evident from the Compoſition of Equations: From which likewiſe it is manifeſt, that the Number of Imaginary Roots in any Equation, (not including Imaginary Terms) is even.

1. If you Multiply each Term in this Original Quadratick Equation, by the Index of a in that Term, and Divide the Sum of the ſeveral Products by a, the Quotient will be

$$2a \begin{array}{l} -\overline{+b-p} \\ -\overline{+b-q} \end{array} \pm 0.$$

Now if for a in this Quotient, you Subſtitute any real Quantity greater than the greateſt Value of a in the foregoing Equation, that is greater than $b-p$, as b (for tho' b were Determin'd, $b-p$ would be $=$ any Quantity leſs than b, p being Indetermin'd only as being $\sqsubset 0$) you will have $2b \begin{array}{l} -\overline{+b-p} \\ -\overline{+b-q} \end{array}$, that is $p+q$, which is manifeſtly $\sqsubset 0$, ſince p and q are each $\sqsubset 0$.

2. If in the foregoing Original Quadratick Æquation, you likewiſe Subſtitute b for a, and from that Sum Subſtract the Product of $p+q$ Multiplyed by the Exceſs of b, above the greateſt Value of a in the ſaid Equation, that is by p; the Remainder will be

$$b^2 \begin{array}{l} -\overline{b-p} \\ -\overline{b-q} \end{array} b + b^2 \begin{array}{l} -p \\ -q \end{array} b + pq - p^2 - pq, = -pp,$$

which is manifeſtly $\sqsupset 0$, ſince the Value of p is not Imaginary.

☞ But here Obſerve, that tho' $-\sqrt{-s}$ be $\sqsubset 0$, yet $-\sqrt{-s} \times -\sqrt{-s} = -s$ is $\sqsupset 0$: and tho' $1-\sqrt{-1}$ be $\sqsubset 0$, yet $\overline{1-\sqrt{-1}}|^4 = -4$ is $\sqsupset 0$.

II. Again, Since any Original Cubick Equation may alſo be Deſign'd by the following one, *viz.*

$$a^3 \begin{array}{l} -\overline{+b-p} \\ -\overline{+b-q} \\ -\overline{+b-r}. \end{array} a^2 \begin{array}{l} +\overline{b^2 \begin{smallmatrix} -p \\ -q \end{smallmatrix} b + pq} \\ +\overline{b^2 \begin{smallmatrix} -q \\ -r \end{smallmatrix} b + pr} \\ +\overline{b^2 \begin{smallmatrix} -q \\ -r \end{smallmatrix} b + qr} \end{array} a$$

$-+b^3$

$$\overline{\quad - + b^3 \begin{array}{l}-p\\-q\\-r\end{array} b^2 \begin{array}{l}+pq\\+pr\\+qr\end{array} b - pqr} = 0.$$

In which Equation, the three Roots are $b-p$, $b-q$ and $b-r$; and ſuppoſe p not $\sqsubset$ q or r, and $\sqsubset 0$, then $b-p$ will be $=$ the greateſt Value of a; which, and each Member thereof, ſuppoſe to be $=$ real Quantities.

1. If you Multiply each Term in this Equation, by the Index of a in that Term, and Divide the Sum of the ſeveral Products by a, the Quotient will be

$$3a^2 \begin{array}{l} -2\times\overline{b-p} \\ -2\times\overline{b-q} \\ -2\times\overline{b-r} \end{array} a \quad \begin{array}{l} +\overline{b^2 \begin{array}{l}-p\\-q\end{array} b + pq} \\ +\overline{b^2 \begin{array}{l}-p\\-r\end{array} b + pr} \\ +\overline{b^2 \begin{array}{l}-q\\-r\end{array} b + qr} \end{array}$$

Now if for a in this Quotient, you Subſtitute any Quantity greater than $b-p$ as b, you'll have, after due Reduction, $pq+pr+qr$, which is evidently $\sqsubset 0$, if q and then of conſequence r be $=$ real Quantities; if not, ſuppoſe $q = y-\sqrt{-z}$, and then r will be $= y+\sqrt{-z}$, and $pq+pr+qr$ will become $= p\times\overline{y-\sqrt{-z}} + p\times\overline{y+\sqrt{-z}} + \overline{y-\sqrt{-z}}\times\overline{y+\sqrt{-z}} = 2p\times y + y^2 + z$, which is likewiſe evidently $\sqsubset 0$.

2. If in the foregoing Original Cubick Equation, you likewiſe Subſtitute b for a, and from that Sum Subſtract $\overline{pq+pr+qr}\times p$, you'll have, after due Reduction, $-p^2q-p^2r$, which is manifeſtly $\sqsupset 0$, if q and r be $=$ real Quantities; if not, by Subſtituting the above Imaginary Values of q and r for them, $-p^2q-p^2r$, will become $= -p^2\times\overline{y-\sqrt{-z}} - p^2\times\overline{y+\sqrt{-z}} = -2p^2y$, which is likewiſe manifeſtly $\sqsupset 0$.

III. *Again*, Since any Original Biquadratick Equation may be Deſign'd by the following one, *viz.*

$$a^4\quad\begin{array}{l} = \overline{+b-p}\\ -\overline{+b-q}\\ -\overline{+b-r}\\ -\overline{+b-s}\end{array}\quad a^3$$

$$\begin{array}{l}\overline{+b^2 \begin{smallmatrix}-p\\-q\end{smallmatrix} b+pq}\\ \overline{+b^2 \begin{smallmatrix}-p\\-r\end{smallmatrix} b+pr}\\ \overline{+b^2 \begin{smallmatrix}-p\\-s\end{smallmatrix} b+ps}\\ \overline{+b^2 \begin{smallmatrix}-q\\-r\end{smallmatrix} b+qr}\\ \overline{+b^2 \begin{smallmatrix}-q\\-s\end{smallmatrix} b+qs}\\ \overline{+b^2 \begin{smallmatrix}-r\\-s\end{smallmatrix} b+rs}\end{array}\quad a^2$$

$$\left.\begin{array}{l}-\overline{+b^3 \begin{smallmatrix}-p\\-q\\-r\end{smallmatrix} b^2 \begin{smallmatrix}+pq\\+pr\\+qr\end{smallmatrix} b-pqr}\\ -\overline{+b^3 \begin{smallmatrix}-p\\-q\\-s\end{smallmatrix} b^2 \begin{smallmatrix}+pq\\+ps\\+qs\end{smallmatrix} b-pqs}\\ -\overline{+b^3 \begin{smallmatrix}-p\\-r\\-s\end{smallmatrix} b^2 \begin{smallmatrix}+pr\\+ps\\+rs\end{smallmatrix} b-prs}\\ -\overline{+b^3 \begin{smallmatrix}-q\\-r\\-s\end{smallmatrix} b^2 \begin{smallmatrix}+qr\\+qs\\+rs\end{smallmatrix} b-qrs}\\ \overline{+b^4 \begin{smallmatrix}-p\\-q\\-r\\-s\end{smallmatrix} b^3 \begin{smallmatrix}+pq\\+pr\\+ps\\+qr\\+qs\\+rs\end{smallmatrix} b^2 \begin{smallmatrix}-pqr\\-pqs\\-prs\\-qrs\end{smallmatrix} b+pqrs.}\end{array}\right\} = 0.\quad a$$

In which Equation the four Values of a, are $b-p$, $b-q$, $b-r$, and $b-s$; and ſuppoſe p not $\sqsubset$ q or r or s, and $\sqsubset$ 0; then $b-p$ will be = the greateſt Value of a, which, and each Member of $b-p$ ſuppoſe to be = real Quantities, then if q and r be = real Quantities, s will be ſo too, but if the Value of q be imaginary, that of r or s will be ſo too.

1. If you Multiply each Term in this Equation, by the Index of a in that Term, and Divide the Sum of the ſeveral Products by a, the Quotient will be

$$4a^3 \begin{matrix} -3\times\overline{b-p} \\ -3\times\overline{b-q} \\ -3\times\overline{b-r} \\ -3\times\overline{b-s} \end{matrix}\; a^2 \quad \begin{matrix} +2\times\overline{b^2\begin{smallmatrix}-p\\-q\end{smallmatrix}b+pq} \\ +2\times\overline{b^2\begin{smallmatrix}-p\\-r\end{smallmatrix}b+pr} \\ +2\times\overline{b^2\begin{smallmatrix}-p\\-s\end{smallmatrix}b+ps} \\ +2\times\overline{b^2\begin{smallmatrix}-q\\-r\end{smallmatrix}b+qr} \\ +2\times\overline{b^2\begin{smallmatrix}-q\\-s\end{smallmatrix}b+qs} \\ +2\times\overline{b^2\begin{smallmatrix}-r\\-s\end{smallmatrix}b+rs} \end{matrix}\; a$$

$$\begin{matrix} -\overline{+b^3\begin{matrix}-p\\-q\\-r\end{matrix}b^2\begin{matrix}+pq\\+pr\\+qr\end{matrix}b-pqr} \\ -\overline{+b^3\begin{matrix}-p\\-q\\-s\end{matrix}b^2\begin{matrix}+pq\\+ps\\+qs\end{matrix}b-pqs} \\ -\overline{+b^3\begin{matrix}-p\\-r\\-s\end{matrix}b^2\begin{matrix}+pr\\+ps\\+rs\end{matrix}b-prs} \\ -\overline{+b^3\begin{matrix}-q\\-r\\-s\end{matrix}b^2\begin{matrix}+qr\\+qs\\+rs\end{matrix}b-qrs} \end{matrix}$$

Now if for a in this Quotient, you Subſtitute any Quantity greater than $b-p$ as b, you'll have, after due Reduction, $pqr+pqs+prs+qrs$, which is plainly $\sqsubset 0$ if q, r and s be $=$ real Quantities: if not, ſuppoſe q and r to be $=$ the two Imaginary ones, and $=y-\sqrt{-z}$ and $y+\sqrt{-z}$ reſpectively, and then s will be $=$ real Quantity; and $pqr+pqs+prs+qrs$, will become $=p\times\overline{y-\sqrt{-z}}\times\overline{y+\sqrt{-z}}+p\times\overline{y-\sqrt{-z}}\times s+p\times\overline{y+\sqrt{-z}}\times s+\overline{y-\sqrt{-z}}\times\overline{y+\sqrt{-z}}\times s=py^2+pz+2pys+y^2s+zs$, which is likewiſe plainly $\sqsubset 0$.

2. If in the foregoing Original Biquadratick Equation, you likewiſe Subſtitute b for a, and from that Sum Subſtract $\overline{pqr + pqs + prs + qrs} \times p$, the Remainder will be found, after due Reduction, $= - p^2qr - p^2qs - p^2rs$, which is manifeſtly $\sqsupset 0$, if q, r and s be $=$ real Quantities; if not, then q and r being Deſign'd as above, $- p^2qr - p^2qs - p^2rs$ will become $= - p^2 \times \overline{y^2 + z} - p^2s \times 2y$, which is likewiſe manifeſtly $\sqsupset 0$.

Corollary.

Now the Sum of what is ſaid in theſe three Paragraphs, and what I deſign to Deduce from them is,

1. That in any Æquation whatſoever, where the greateſt Value of the unknown Root $= a$ is not Imaginary, and the firſt Term or higheſt Power of the ſaid Root a is Affirmative, and the laſt Term an abſolute Number or given Quantity; if you Multiply each Term by the Index of a in that Term, and Divide the Sum of the ſeveral Products by a, and then inſtead of a in the Quotient Subſtitute any Number or Quantity $= b$ greater by any real Quantity than the greateſt Value of a the Sum of this Quantity, which here Note, will be $\sqsubset 0$.

2. And if in the ſaid Æquation, inſtead of a you Subſtitute b, and from that Sum Subſtract the above Noted Quantity $\times \overline{b - a}$ the Remainder will be $\sqsupset 0$.

☞ *See the Uſe of this Corollary, in Demonſtrating the Converging Series.*

PART X.

The Solution of Adfected, Quadratick, Æquations.

All Adfected Quadratick Æquations, do (as hath been already ſhewn) fall under the Conſideration of theſe four Forms or Caſes, *viz*.

$$\left.\begin{array}{l} a^2 + pa - b = 0 \\ a^2 - pa - b = 0 \\ a^2 - qa + b = 0 \\ a^2 + qa + b = 0 \end{array}\right\} \text{Or Univerſally} \left\{\begin{array}{l} a^n + pa^{\frac{n}{2}} - b = 0 \\ a^n - pa^{\frac{n}{2}} - b = 0 \\ a^n - qa^{\frac{n}{2}} + b = 0 \\ a^n + qa^{\frac{n}{2}} + b = 0 \end{array}\right.$$

When there happens to be more Terms in one of theſe kind of Æquations than two, and the higheſt Power of the unknown Quantity is Multiplyed into ſome known Coefficient, you muſt reduce them by Diviſion (as in Part VII. *Sect.* 4) and for the Fractional Quantities that may ariſe by thoſe Diviſions, Subſtitute another whole Quantity.

For Inſtance, Let $baa + caa - da - ca = dc + cb$.

Then by Dividing by $b + c$, you'll have

$$aa - \frac{\overline{d + c} \times a}{b + c} = \frac{dc + cb}{b + c}.$$

Now make $\frac{d + c}{b + c} = k$.

And for $\frac{dc + cb}{b + c}$, put l.

Then will $aa - ka = l$, be the new Æquation equal to the other, and now fitted for a Solution. But here we will leave it, and return to our four foregoing Caſes;

In order to ſolve each of which, it will be requiſite to premiſe this

Lemma.

Half (d) the Difference of any two Quantities (z and y, whereof z is the greater) Added to half (s) their Sum, is equal to the Greater; but Subſtracted from half their Sum, is equal to the Leſſer of them.

Demonſtration.

$$\begin{array}{r|c|l}
 & 1 & \frac{1}{2}z + \frac{1}{2}y = \frac{1}{2}s \\
 & 2 & \frac{1}{2}z - \frac{1}{2}y = \frac{1}{2}d \\
1 + 2 & 3 & z = \frac{1}{2}s + \frac{1}{2}d \\
1 - 2 & 4 & y = \frac{1}{2}s - \frac{1}{2}d.
\end{array}$$

Q. E. D.

It hath been already * prov'd, that in each of the four foregoing Forms or Caſes of Adfected Quadratick Æquations, the Sum (s) and Rectangle (r) of the two ſought Roots, are given to find their Difference, and then the Roots themſelves ſeverally.

Now (by the foregoing *Lemma*)

$\frac{1}{2}s + \frac{1}{2}d =$ greater of the ſought Roots.

And $\frac{1}{2}s - \frac{1}{2}d =$ leſſer of them.

And by Multiplying them by one another, you'll have $\frac{1}{4}ss - \frac{1}{4}dd = r$.

Then by Multiplying the laſt Æquation by 4, you'll have $ss - dd = 4r$.

That is, by Tranſpoſition $ss - 4r = dd$.

And by Evolution $\sqrt{ss - 4r} = d$.

And by our *Lemma*

$$\frac{1}{2}s + \frac{\sqrt{ss - 4r}}{2} = \text{greater of the ſought Roots.}$$

$$\frac{1}{2}s - \frac{\sqrt{ss - 4r}}{2} = \text{leſſer of them.}$$

Wherefore the Canon for finding the two Values of a in this 1*ſt* Caſe, *viz.*

$aa + pa - b = 0$ is (becauſe, as we have already prov'd, s is $= -p$, and $r = -b$ in this 1*ſt* Caſe) $-\frac{1}{2}p \pm \frac{\sqrt{pp + 4b}}{2}$

$$= \left\{\begin{array}{l}\text{Greater}\\ \text{Leſs or Negative}\end{array}\right\} \text{Value of } a.$$

Alſo the *Canon* for finding the two Values of a in this 2*d* Caſe, *viz*, $aa - pa - b = 0$ is (becauſe s is $= p$ and $r = -b$ in

aa+4a=60 pons a=x-2 hinc
xx-4x+4+4x-8=60 = xx-4 hinc xx=64 x=8
a=6 36+24=60

* See Page 114.

in this 2*d* Caſe) $\frac{1}{2}p \pm \frac{\sqrt{pp + 4b}}{2} = \left\{\begin{array}{l}\text{Greater}\\ \text{Leſs or Negative}\end{array}\right\}$ Value of *a*.

Again, the *Canon* for finding the two Values of *a* in this 3*d* Form or Caſe, *viz.* $aa - qa + b = 0$ is (becauſe s is $= q$ and $r = b$ in this 3*d* Caſe),

$\frac{1}{2}q \pm \frac{\sqrt{qq - 4b}}{2} = \left\{\begin{array}{l}\text{Greater}\\ \text{Leſs}\end{array}\right\}$ Value of *a*; both of which Values are Affirmative if $qq \sqsubset 4b$; and for that Reaſon this Æquation is call'd Ambiguous; but if $qq \sqsupset 4b$, then the ſaid Values are Imaginary: for, in this latter Caſe, the Square-Root of a Negative Quantity is to be Extracted in order to find the Value of *a*. But a Negative Quantity hath no Square-Root; for whether the Root be Affirmative or Negative, its Square will be Affirmative; wherefore ſuch impoſſible Æquations are ſaid to have Imaginary Roots.

And the *Canon* for finding the Values of *a* in this 4*th* Caſe *viz.* $aa + qa + b = 0$ is (becauſe $s = -q$ and $r = b$ in this 4*th* Caſe) $-\frac{1}{2}q \pm \frac{\sqrt{qq - 4b}}{2} = \left\{\begin{array}{l}\text{Greater}\\ \text{Leſs}\end{array}\right\}$ Value of *a*, both of which Values are either Negative or Imaginary, and therefore this laſt Caſe is not taken notice of by many Algebraiſts. And I (for Brevities ſake) will inſert it no more.

Tho' the foregoing Method of reſolving Adfected Quadratick Æquations, is what naturally follows from Mr. *Harriot*'s Method of Compounding them; yet it is not the ſame with his, which is a peculiar way of his own: and that is, after Tranſpoſing *b*, by Adding the Square of half the Coefficient, to each part of the Æquation, he thereby makes the unknown Part a compleat Square in Species.

Thus by Tranſpoſing the known or given Quantity *b* in the three firſt foregoing Forms or Caſes, they will become

$$\left.\begin{array}{l}1.\ aa + pa = b\\ 2.\ aa - pa = b\\ 3.\ aa + qa = -b\end{array}\right\}\text{Reſpectively.}$$

And theſe Mr. *Harriot* calls Canonical Æquations.

Now 'tis manifeſt that if you Add the Square of half the Coefficient to both ſides of each Æquation, the firſt Part of it will thereby become a perfect Square, by which Means the Value of the ſought Root *a* will be made known; thus

1. In the 1*ſt* Form where $aa + pa = b$,

Add the Square of $\frac{1}{2}p$ towit $\frac{1}{4}pp$ to each Part, and the Sums are $aa + pa + \frac{1}{4}pp = b + \frac{1}{4}pp$.

$aa - 4a = 12$ $a = x + 2$ $[a = 6$ $xx = 16$ $x = 4$ Then

$xx + 4x + 4 - 4x - 8 = 12 = xx - 4$

Then by Extracting the Square-Root of each Part,

$a + \frac{1}{2} p = \pm \sqrt{b + \frac{1}{4} pp}$

And, by Tranſpoſing $\frac{1}{2} p$ you have

eor. 1. $a = - \frac{1}{2} p \pm \sqrt{b + \frac{1}{4} pp}$, as before.

2. In the 2*d* Form, where $aa - pa = b$.

By Compleating □ or Adding the Square of $\frac{1}{2} p$ to each Part,

$aa - pa + \frac{1}{4} pp = b + \frac{1}{4} pp$.

Then by Extracting the Square-Root of each Part,

eor. 2. $a - \frac{1}{2} p = \pm \sqrt{b + \frac{1}{4} pp}$.

And by Tranſpoſition $a = \frac{1}{2} p \pm \sqrt{b + \frac{1}{4} pp}$, as before.

3. In the 3*d* Caſe or Form, where $aa - qa = - b$.

Compleat the Square, by Adding the Square of half the Coefficient $- q$, that is $\frac{1}{4} qq$ to each Part, and you'll have

$aa - qa + \frac{1}{4} qq = - b + \frac{1}{4} qq$.

Then by Extracting the Square-Root of each Part

$a - \frac{1}{2} q = \pm \sqrt{- b + \frac{1}{4} qq}$.

And by Tranſpoſition

Teor. 3. $a = \frac{1}{2} q \pm \sqrt{- b + \frac{1}{4} qq}$, as before.

Now by the help of theſe Theorems it will be eaſy to Calculate or find the Value of the unknown Quantity (a) in Numbers.

Example 1. *Caſe* 1.

Suppoſe $aa + a = 6$;

Then $p = 1$, and $b = 6$; wherefore $a = - \frac{1}{2} \pm \sqrt{6 \frac{1}{4}}$ by Theorem 1; that is $a = - \frac{1}{2} \pm 2 \frac{1}{2} = \left\{ \begin{matrix} + 2 \\ - 3 \end{matrix} \right\}$; that is to ſay, the two Values of a are $+ 2$ and $- 3$. And either of theſe Roots being known, the other may be found by Subſtracting the known one, from the Coefficient of the 2*d* Term with a contrary Sign, as is evident from the Compoſition of adfected Quadraticks.

So, if I know that $+ 2$ is one of the Roots of the above Æquation $a^2 + a = 6$, I can have the other $- 3$ by Subſtracting $+ 2$ from the Coefficient of the 2*d* Term with a contrary Sign, that is from $- 1$: or if I had known that $- 3$ is one of the Roots (or Values of a) in the ſaid Æquation, I can find the other $+ 2$, by Subſtracting $- 3$ from $- 1$: or if one of the Roots be known, the other may be had by reducing all the Terms in the given Æquation to one Side, and Æquating them to 0, and then Dividing them by a — given (or known) Quantity $= 0$; thus

$$\begin{array}{l} a - 2 = 0)\ aa + a - 6 = 0\ (a + 3 = 0 \\ \qquad\qquad\quad \underline{aa - 2a\quad} \\ \qquad\qquad\qquad \ + 3a - 6 \\ \qquad\qquad\qquad\quad \underline{3a - 6} \\ \qquad\qquad\qquad\qquad\ 0 \end{array}$$

$$\begin{array}{l} \text{Or } a + 3 = 0)\ aa + a - 6 = 0\ (a - 2 = 0 \\ \qquad\qquad\qquad \underline{aa + 3a\quad} \\ \qquad\qquad\qquad\quad - 2a - 6 \\ \qquad\qquad\qquad\quad \underline{- 2a - 6} \\ \qquad\qquad\qquad\qquad\ 0 \end{array}$$

☞ *The truth of this is manifeſt from the Compoſition of Original Adfected Quadraticks.*

And tho' it may ſeem wonderful that the Value of a ſhould be -3 in the foregoing Æquation; yet by that Value the true (or firſt) Æquation may be found thus.

Let	1	$a = -3$
1 ⊙ 2	2	$aa = 9$, viz: $= -3 \times -3$
1 + 2	3	$aa + a = 9 - 3 = 6$, as at firſt.

Example 2. *Caſe* 2.

Suppoſe $aa - 7a = 948.75$. Then $-p = -7$ and $b = 948.75$. And by *Canon* (or *Theorem*) 2.

$$a = \frac{7}{2} \pm \sqrt{948.75 + \frac{49}{4}} = \frac{7}{2} \pm \sqrt{961} = 3\frac{1}{2} \pm 31 = \left\{\begin{array}{r} 34.5 \\ -27.5 \end{array}\right\}.$$

Now, either of thoſe Values of a being known, the other may be had by Subſtraction or Diviſion, as in *Example* 1.

Example 3. *Caſe* 3.

Suppoſe $aa - 36a = -243$. Here $-q = -36$, and $-b = -243$.

Then $a = 18 \pm \sqrt{324 - 243}$ by *Canon* 3; that is $a = 18 \pm \sqrt{81} = \left\{\begin{matrix}27\\9\end{matrix}\right\}$.

This 3*d* Form is call'd Ambiguous, becauſe it hath two Affirmative Values of the unknown Quantity (a), either of which Values being had, the other may be found by Subſtraction or Diviſion, in like manner, as in *Example* 1.

Notwithſtanding all Quadratick Æquations of this 3*d* Form have two Affirmative, (or Imaginary) Roots; yet but one of thoſe Roots will give a true Anſwer to the Queſtion, and that is to be choſen according to the Nature and Limits of the Queſtion, *as ſhall be ſhew'd further on.*

From the foregoing *Canons*, by the help of *Subſtitution*, other *Canons* may be Deduc'd, which will ſolve all Affected Quadratick Æquations. Thus

$$\text{If}\ \left|\begin{array}{l|l} 1 & a^n + pa^{\frac{n}{2}} = b \\ 2 & a^n - pa^{\frac{n}{2}} = b \\ 3 & a^n - qa^{\frac{n}{2}} = -b \end{array}\right\}\ \text{Then}$$

Suppoſe $a^{\frac{n}{2}} = e$, and the foregoing Æquations will become

$$\left|\begin{array}{l|l} 1 & ee + pe = b \\ 2 & ee - pe = b \\ 3 & ee - qe = -b \end{array}\right\}\ \text{Reſpectively.}$$

Therefore

$$\text{By Canon}\ \left|\begin{array}{l|l} 1 & e = -\frac{1}{2}p \pm \sqrt{\frac{1}{4}pp + b} \\ 2 & e = \frac{1}{2}p \pm \sqrt{\frac{1}{4}pp + b} \\ 3 & e = \frac{1}{2}q \pm \sqrt{\frac{1}{4}qq - b} \end{array}\right\}\ \text{Reſpectively.}$$

But $e = a^{\frac{n}{2}}$ by Suppoſition; Conſequently

$$\left|\begin{array}{l|l} 1 & a = \sqrt[\frac{n}{2}]{-\frac{1}{2}p \pm \sqrt{\frac{1}{4}pp + b}} \\ 2 & a = \sqrt[\frac{n}{2}]{\frac{1}{2}p \pm \sqrt{\frac{1}{4}pp + b}} \\ 3 & a = \sqrt[\frac{n}{2}]{\frac{1}{2}q \pm \sqrt{\frac{1}{4}qq - b}} \end{array}\right\}\ \text{Reſpectively.}$$

There is another Method of reſolving Adfected Quadratick Æquations, and that is by caſting off the 2*d* or loweſt Term of the unknown Quantity, which is done by *Subſtitution*, thus; take always half the known Coefficient, and in Caſes 1*ſt* and 4*th* add it to, but in Caſes 2*d* and 3*d* Subſtract it from, its fellow Factor, and for their Sum or Difference Subſtitute another Letter. As in theſe

Caſe 1.

Let	1	$a^n + pa^{\frac{n}{2}} = b$
Put	2	$a^{\frac{n}{2}} + \frac{1}{2}p = e$
2 ⊙ 2	3	$a^n + pa^{\frac{n}{2}} + \frac{pp}{4} = ee$
3 — 1	4	$\frac{1}{4}pp = ee - b$
4 + *b*	5	$\frac{1}{4}pp + b = ee$
5 ɯ 2	6	$\sqrt{\frac{1}{4}pp + b} = e$
From 2 and 6	7	$a^{\frac{n}{2}} + \frac{1}{2}p\,(= e) = \sqrt{\frac{1}{4}pp + b}$
7 — ½ *p*	8	$a^{\frac{n}{2}} = -\frac{1}{2}p + \sqrt{\frac{1}{4}pp + b}$
8 ɯ $\frac{n}{2}$	9	$a = \sqrt[\frac{n}{2}]{-\frac{1}{2}p + \sqrt{\frac{1}{4}pp + b}}$

Caſe 2.

Let	1	$a^n - pa^{\frac{n}{2}} = b$
Put	2	$a^{\frac{n}{2}} - \frac{1}{2}p = e$
2 ⊙ 2	3	$a^n - pa^{\frac{n}{2}} + \frac{1}{4}pp = ee$
3 — 1	4	$\frac{1}{4}pp = ee - b$
4 + *b*	5	$\frac{1}{4}pp + b = ee$
5 ɯ 2	6	$\sqrt{\frac{1}{4}pp + b} = e$
From 2 and 6	7	$a^{\frac{n}{2}} - \frac{1}{2}p\,(= e) = \sqrt{\frac{1}{4}pp + b}$
7 + ½ *p*	8	$a^{\frac{n}{2}} = \frac{1}{2}p + \sqrt{\frac{1}{4}pp + b}$
8 ɯ $\frac{n}{2}$	9	$a = \sqrt[\frac{n}{2}]{\frac{1}{2}p + \sqrt{\frac{1}{4}pp + b}}$

Caſe 3.

Let	1	$a^{n} - qa^{\frac{n}{2}} = -b$
Put	2	$a^{\frac{n}{2}} - \frac{1}{2}q = e$
2 ⊙ 2	3	$a^{n} - qa^{\frac{n}{2}} + \frac{qq}{4} = ee$
3 — 1	4	$\frac{1}{4}qq = ee + b$
4 — b	5	$\frac{1}{4}qq - b = ee$
5 ш 2	6	$\pm\sqrt{\frac{1}{4}qq - b} = e$
From 2 and 6	7	$a^{\frac{n}{2}} - \frac{1}{2}q\,(=e) = \pm\sqrt{\frac{1}{4}qq - b}$
7 + $\frac{1}{2}q$	8	$a^{\frac{n}{2}} = \frac{1}{2}q \pm \sqrt{\frac{1}{4}qq - b}$
8 ш $\frac{n}{2}$	9	$a = \sqrt[\frac{n}{2}]{\frac{1}{2}q \pm \sqrt{\frac{1}{4}qq - b}}$

Corollary.

Hence it is evident that whatſoever Method is uſed in ſolving theſe (or indeed any other) Æquations, the Reſult will ſtill be the ſame, if the Work be true.

PART

PART XI.

Of Numerical Queſtions.

CHAP. I.

Queſtions producing Simple Æquations.

Queſtion 1.

s (200) Pounds is ſo to be Divided between two Men, that the one is to have *d* (73) Pounds more than the other. What is the Share of each Man?

	1	Suppoſe the Shares ſought to be = a the Greater, and e the Leſſer.
Then	2	$a+e=s$ } by the Queſtion.
	3	$a-e=d$
$2+3$	4	$2a=s+d$
$4\div 2$	5	$a=\frac{s+d}{2}\left(=\frac{200+73}{2}\text{Pounds}\right)=£136:10:00)$ = greater Share.
$2-3$	6	$2e=s-d$
$6\div 2$	7	$e=\frac{s-d}{2}$ = leſſer Share (= 63 *l.* 10 *s.* 00 *d.*)

Or thus,

	1	Suppoſe the greater Share to be = a.
	2	Then the leſſer Share is = $a-d$ } by the Queſtion.
	3	And $a+a-d=s=2a-d$
$3+d$	4	$s+d=2a$
$4\div 2$	5	$\frac{s+d}{2}=a$
And	6	$a-d=\frac{s+d}{2}-d$ = leſſer Share = $\frac{s-d}{2}$

Queſtion 2.

	1	$a + e = s\ (1)$
	2	$a \div e = q\ (100000)$; Quere a and e.
$1 - e$	3	$a = s - e$
$2 \times e$	4	$a = eq$
$3 = 4$	5	$s - e\ (= a) = eq$
$5 + e$	6	$s = eq + e$
$6 \div q + 1$	7	$\frac{s}{q+1} = e\ (= \frac{1}{100000+1})$
From 3 and 7	8	$a = s - \frac{s}{q+1} = \frac{qs}{q+1} (= \frac{100000}{100000+1})$

Queſtion 3.

A Gentleman finding divers Poor People at his Door, gave each of them b (3) Pence, and had c (6) Pence remaining; but, if he wou'd have given them d (4) Pence each, he wou'd want f (2) Pence. How many Poor People were there?

	1	Suppoſe a = Number of Poor People that were there.
Then	2	$ba + c = da - f$ By the Queſtion.
$2 + f$	3	$ba + c + f = da$
$3 - ba$	4	$c + f = da - ba$
$4 \div d - b$	5	$\frac{c+f}{d-b} = a\ (= \frac{6+2}{4-3} = 8$ Poor People.)

Queſtion 4.

One being ask'd how old he was, anſwer'd, if $\frac{c}{b}$ ($\frac{5}{10}$) of my Age in Years, were Multiplyed by $\frac{p}{q}$ ($\frac{1}{8}$) of my Age, the Product wou'd be my Age. I demand his Age?

	1	Suppoſe a = his Age.
Then	2	$\frac{ca}{b} \times \frac{pa}{q} = a = \frac{cpaa}{bq}$ by the Queſtion.
$2 \times bq$	3	$bqa = cpaa$
$3 \div pca$	4	$\frac{bqa}{pca} = a = \frac{bq}{cp}\ (= \frac{10 \times 8}{5 \times 1} = \frac{80}{5} = 16$ Years.

M4 7.

Queſtion 5.

A Man and his Wife did uſually drink a Veſſel of Beer in b (12) Days, and they found, by often Experience, that the Wife being abſent, the Man drank it in c (20) Days; the Queſtion is in how many Days wou'd the Wife alone drink it at that Rate.

1. Suppoſe ſhe wou'd Drink it in a Days.

2. Now in order to find what Quantity he wou'd Drink in a Days, ſay

c Days .. 1 Veſſel :: a Days .. $\frac{a}{c}$ Veſſels the Quantity he wou'd Drink in a Days.

3. Therefore the Quantity (or Number of Veſſels) which he and ſhe wou'd Drink in a Days is $= 1 + \frac{a}{c} = \frac{c+a}{c}$ Veſſels.

4. But he and ſhe did alſo Drink (by the Queſtion) 1 Veſſel in b Days, therefore to find what Quantity they both did Drink in a Days at this Rate, ſay

b Days .. 1 Veſſel :: a Days .. $\frac{a}{b}$ Veſſels.

$3 = 4$	5	$\frac{a+c}{c} = \frac{a}{b}$
$5 \times b$	6	$a = \frac{bc+ba}{c}$
$6 \times c$	7	$ca = bc + ba$
$7 - ba$	8	$ca - ba = bc$
$8 \div \overline{c-b}$	9	$a = \frac{bc}{c-b}$ $\left(= \frac{12 \times 20}{20 - 12} = 30\right.$ Days.)

Queſtion 6.

A Hare being b (50) Paces, or Leaps of her own, before a Grey-hound, makes r (4) Leaps to the Greyhounds s (3) Leaps. But m (2) Leaps of the Grey-hounds are as much as n (3) Leaps of the Hares. How many Leaps muſt the Grey-hound take to Catch the Hare?

| 1 | For the Number of the Gray-hounds Leaps ſought put a.

Then

Then	2	$s\,..\,r :: a\,..\,\frac{ra}{s}$	*That is, the Number of the Gray-hound's Leaps is to them of the Hares in any time, ſo is all the Gray-hounds Leaps to all the Hares Leaps after the Gray-hound began to Courſe her.*
	3	And $\frac{ra}{s} + b = \frac{ra + sb}{s} =$ to the whole Number of Paces the Hare went.	
	4	$m\,..\,n :: a\,..\,\frac{ra + bs}{s}$	*That is m is to n as the whole Number of the Gray-hounds Leaps is to the whole Number of the Hares Leaps.*
Therefore	5	$\frac{mra + mbs}{s} = na$	
$5 \times s$	6	$mra + mbs = sna$	
$6 - mra$	7	$mbs = sna - mra$	
$7 \div \overline{sn - mr}$	8	$\frac{mbs}{sn - mr} = a$ $\left(= \frac{2 \times 50 \times 3}{3 \times 3 - 2 \times 4} = 300\right.$ Leaps of the Gray-hounds.)	

Or, having proceeded as far as the 3*d* Step, it is evident that a Leaps of the Gray-hounds muſt be equal to $\frac{ra + sb}{s}$ Leaps of the Hare's, and by the latter Part of the Queſtion, m Leaps of the Gray-hound's is equal to n Leaps of the Hare's; Therefore ma Leaps of the Gray-hound's is equal to na Leaps of the Hare's, and (by Dividing each Part by m) a Leaps of the Gray-hound's is equal $\frac{n}{m}a$ Leaps of the Hare's, which is (by what has been here ſaid) $= \frac{sb + ra}{s}$. And by Multiplying each Part by m, you get the Æquation in the 5*th* Step foregoing.

Vitruvius (in *Lib.* 9. *Cap.* 3.) reporteth that King *Hiero* having given Commandment for the making of a Crown of Pure Gold, was inform'd that the Workman had detain'd part of the Gold, and mixt the reſt with as much Silver as he had ſtoln of Gold. The King being much diſpleas'd at the Deceit, recommended the Examination of the Buſineſs to the famous *Archimedes* of *Syracuſe*, who without defacing the Crown, diſcover'd the Cheat in this manner, *viz*. Experience telling him that a Quantity of Gold would poſſeſs leſs Room, or Space, than a Quantity of Silver of the ſame Weight, and conſequently that a mixt Maſs of Gold and Silver of the ſame Weight, wou'd take up ſome mean Space between the two former, He made a Maſs of pure Gold of the ſame Weight with the Crown; likewiſe another Maſs of Silver of the ſame Weight with the Crown. Then having put the Crown, as alſo the other two Maſſes ſeverally into a Veſſel fill'd up to the Brim with Water, he diligently reſerv'd the Water flowing over into another Veſſel, and from thoſe three Quantities of Water ſo expell'd, he found out the Quantity of Gold and Silver in the Crown; but ſince *Vitruvius* delivers not the Practical Operation, I ſhall here ſhew the ſame after the manner of *Cardanus* and others.

Queſtion 7.

Suppoſe the Weight of the Crown, as alſo of the two Maſſes to have been each $= b$ ℔: alſo that by putting in the Maſs of Gold into the Veſſel, that c ℔ of Water was thereby expell'd, by putting in the Maſs of Silver d ℔, and by putting in the Crown f ℔. The Queſtion is to know how much Gold, and how much Silver the Crown was Compoſ'd of.

Suppoſe of a ℔ of Gold, and therefore of $b - a$ ℔ of Silver; Then

If b ℔ of Gold expel .. c ℔ of Water : : a ℔ of Gold will expel .: $\frac{ca}{b}$ ℔ of Water.

Again if b ℔ of Silver expel .. d ℔ of Water : : $b - a$ ℔ of Silver will expel .. $\frac{db - da}{b}$ ℔ of Water; and therefore the Quantity of Gold and Silver in the Crown will expel $\frac{ca + db - da}{b}$ ℔ of Water, which muſt be (by the Queſtion) $= f$ ℔ of Water.

.. And by Multiplying each Part by b, we have $ca - da + db = fb$.

By Tranſpoſition $ca - da = fb - db$.

By Diviſion $a = \frac{fb - db}{c - d}$; and therefore $b - a = b - \frac{fb - db}{c - d}$.

Hæc Excogitatio Archimedis *dici meruit*.

Queſtion 8.

Divide 90 into four ſuch Numbers, that if the firſt be Increaſed with 2; the ſecond leſſened by 2; the third Multiplyed by 2, and the fourth Divided by 2; the Sum, Remainder, Product, and Quotient may be equal between themſelves.

$$\text{Suppoſe } 90 = b$$
$$2 = d.$$

And the four Numbers ſought $= a, e, u$ and y reſpectively. Then

Solution.

	1	$a + e + u + y = b$
And	2	$a + d = e - d$
And	3	$a + d = du$
And	4	$a + d = \frac{y}{d}$
	5	$a = a$
$2 + d$	6	$a + 2d = e$
$3 \div d$	7	$\frac{a + d}{d} = u$
$4 \times d$	8	$ad + dd = y$
$5 + 6 + 7 + 8$	9	$2a + 2d + \frac{a + d}{d} + ad + dd \; (= a + e + u + y) = b$
9 Reduc'd	10	$a = \frac{bd - ddd - 2dd - d}{dd + 2d + 1}$
From 10 and 6	11	$e = \frac{bd + d^3 + 2dd + d}{dd + 2d + 1}$
From 10 and 7	12	$u = \frac{b}{dd + 2d + 1}$
From 10 and 8	13	$y = \frac{bdd}{dd + 2d + 1}$

Which four laſt Æquations give a Canon for finding the four Numbers ſought, which are 18, 22, 10 and 40.

Queſtion 9.

$$a + \frac{e + x + y + z}{3} = b \; (2380)$$
$$e + \frac{a + x + y + z}{4} = c \; (2205)$$
$$x + \frac{a + e + y + z}{5} = d \; (2268)$$
$$y + \frac{a + e + x + z}{6} = f \; (2450)$$

$z +$

And $z + \frac{a + e + x + y}{7} = g$ (2700)

The Queſtion is what the Value of a is.

The Queſtion being reduc'd to whole Quantities gives	1	$3a + e + x + y + z = 3b$
	2	$4e + a + x + y + z = 4c$
	3	$5x + a + e + y + z = 5d$
	4	$6y + a + e + x + z = 6f$
	5	$7z + a + e + x + y = 7g$
$2 - 1$	6	$3e - 2a = 4c - 3b$
$3 - 1$	7	$4x - 2a = 5d - 3b$
$4 - 1$	8	$5y - 2a = 6f - 3b$
$5 - 1$	9	$6z - 2a = 7g - 3b$
$6 + 2a$	10	$3e = 2a + 4c - 3b$
$7 + 2a$	11	$4x = 2a + 5d - 3b$
$8 + 2a$	12	$5y = 2a + 6f - 3b$
$9 + 2a$	13	$6z = 2a + 7g - 3b$
	14	$3a = 3a$
$10 \div \overline{3}$	15	$e = \frac{2a}{3} + \frac{4c - 3b}{3}$
$11 \div \overline{4}$	16	$x = \frac{2a}{4} + \frac{5d - 3b}{4}$
$12 \div \overline{5}$	17	$y = \frac{2a}{5} + \frac{6f - 3b}{5}$
$13 \div \overline{6}$	18	$z = \frac{2a}{6} + \frac{7g - 3b}{6}$
$14 + 15 + 16 + 17 + 18$	19	$3a + e + x + y + z = 3a + \frac{2a}{3} + \frac{2a}{4} + \frac{2a}{5} + \frac{2a}{6} + \frac{4c}{3} + \frac{5d}{4} + \frac{6f}{5} + \frac{7g}{6} - \frac{3b}{3} - \frac{3b}{4} - \frac{3b}{5} - \frac{3b}{6} = 3b$ (by the 1ſt Step).
19 reduc'd	20	$a = \frac{351b - 80c - 75d - 72f - 70g}{294}$ $(= 420)$.

a being thus made known, e, x, y and z will be made known, by the 15*th*, 16*th*, 17*th* and 18*th* Steps, (that is $e = 840$, $x = 1260$, $y = 1680$, and $z = 2100$).

Queſtion 10.

Suppoſe a Clock hath two Indices A and B, and that A is carried b (24) Circumferences (or b times round) in c (24) Hours, and B is carried the ſame way d (21) Circumferences in f (12) Hours. The Queſtion is if they were conjoin'd at this Inſtant, in how many Hours wou'd they be conjoin'd again.

Suppoſe $a =$ Number of Hours required,

Then c Hours .. b Circumferences :: a Hours .. $\frac{ab}{c}$ Circumferences $=$ A's whole Courſe in a Hours.

f Hours .. d Circumferences :: a Hours .. $\frac{ad}{f}$ Circumferences $=$ B's whole Courſe in a Hours.

It is evident that the Index whoſe Motion is ſwifteſt will outgo the ſloweſt 1 Circumference in a Hours; whence

$$\frac{ab}{c} \backsim \frac{ad}{f} = 1$$

That is $\frac{abf \backsim adc}{cf} = 1$

And, by Multiplying each Part by cf, you'll have $abf \backsim adc = cf$.

And by Dividing each Part by $bf \backsim dc$, you'll have $a = \frac{cf}{bf \backsim dc} \left(= \frac{24 \times 12}{24 \times 12 - 1 \times 24} = \frac{288}{264} = 1\,\frac{24}{264} = 1\,\frac{1}{11}\right.$ Hours).

Lemma

To find the leaſt Number, which, being Divided by two or more Numbers, ſhall have no Remainder.

Rule.

Reduce the two firſt taken Numbers to a common Denominator, then Multiply their Numerators by one another, and Multiply the greateſt common Meaſure of the Numerators by their common Denominator; by the laſt Product Divide the Product of the Numerators, and the Quotient is the leaſt Number that can be Divided by the two 1ſt taken Numbers without having a Remainder; which found Number take as one, and one of the remaining Numbers (if there be any) as another, and work with thoſe in like manner as with the two firſt taken Numbers, and ſo proceed with all the other Numbers (if more there be) and the laſt Quotient is the Number required.

In order to demonſtrate this Rule, I will ſuppoſe the two firſt taken Numbers, after they are reduc'd to a common Denominator, to be equal to $\frac{b}{c}$ and $\frac{d}{c}$, alſo g to be equal to the greateſt common Meaſure of the Values of b and d, and $a =$ leaſt Number that can be Divided by $\frac{b}{c}$ and $\frac{d}{c}$ without leaving a Remainder. I ſay that $a = \frac{bd}{cg}$.

Demonſtration.

For, if $\frac{bd}{cg}$ be Divided by $\frac{b}{c}$ and $\frac{d}{c}$ ſeverally, the Quotients will be equal to whole Numbers, to wit to $\frac{d}{g}$ and $\frac{b}{g}$; wherefore if a be not $\sqsupset \frac{bd}{cg}$, it is equal to it. Now in order to Demonſtrate that a is not $\sqsupset \frac{bd}{cg}$, ſuppoſe $a \sqsupset \frac{bd}{cg}$, and ſuppoſe e to be ſuch as that it may be equal to any Number that is $\sqsubset 1$, then any Number whatſoever $\sqsupset \frac{bd}{cg}$ may be Deſign'd by $\frac{bd}{cge}$: And ſuppoſe $a = \frac{bd}{cge}$. Now this Value of a being Divided by $\frac{b}{c}$ and $\frac{d}{c}$ ſeverally gives the two Quotients equal to $\frac{d}{ge}$ and $\frac{b}{ge}$; either, or both of which Quotients is, or are, equal to a Fractional Number or Numbers (which is contrary to what is required), or g is not $=$ the greateſt common Meaſure of b and d, but ge (which is contrary to our ſuppoſition); wherefore the Suppoſition, namely that a is $\sqsupset \frac{bd}{cg}$, is abſurd; and conſequently a is $=$ $\frac{bd}{cg}$.

Hence it is evident, that all the Numbers that can be Divided by the Values of $\frac{b}{c}$ and $\frac{d}{c}$ without having a Remainder are $=$ Multiples of $\frac{bd}{cg}$ as $\frac{bd}{cg}$, $\frac{2bd}{cg}$, $\frac{3bd}{cg}$, $\frac{4bd}{cg}$, &c; So that if you take one (if there be any) of the remaining Numbers and find the leaſt Number that can be meaſur'd by it, and the Value of the before found Quantity $\frac{bd}{cg}$, you will have the leaſt Number that can be Meaſur'd (or Divided without leaving a Remainder) by the Values of $\frac{b}{c}$, $\frac{d}{c}$ and by the laſt taken Number. Finally by proceeding ſo with all the other Numbers (if there Remains more) you will manifeſtly find the leaſt Number that can be Divided by the given Numbers, without leaving a Remainder. *Q. E. D.*

Queſtion 11.

Suppoſe a Clock hath 4 Indices, towit A, B, C and D. A is carried 1 time round the whole Circumference of the ſaid Clock in one Hour; B is carried the ſame way $1\frac{1}{2}$ times round in one Hour; C is carried the ſame way $2\frac{1}{3}$ times round in one Hour; and D is carried the ſame way 3 times round the Circumference of the ſaid Clock in one Hour. The Queſtion is in how many Hours after all the ſaid Indices Commence a Motion from the ſame Point, will they all be conjoin'd again.

Suppoſe A to be conjoin'd with $\left\{\begin{array}{c}B\\C\\D\end{array}\right\}$ in $\left\{\begin{array}{c}a\\e\\y\end{array}\right\}$ Hours.

Since A's Motion is the ſloweſt, it is evident that either of the Reſt, towit B, C, or D will be carried 1 Circumference more than A at the time that it will be conjoin'd with (or overtake) A; wherefore in order to find in how many Hours A and B will be conjoin'd, ſay

$a \times 1 + 1$ Circumferences $= a \times 1\frac{1}{2}$ Circumferences.

That is $a + 1 = 1\frac{1}{2}\,a$

Therefore $1 = 1\frac{1}{2}\,a - a = \frac{1}{2}\,a$

Conſequently $2 = a$; that is to ſay A and B will be conjoin'd in 2 Hours.

In like manner e will be found $= \frac{3}{4}$, which is the Number of Hours wherein the Conjunction of A and C will be accompliſh'd.

And by a like proceſs with the foregoing in finding the Value of a, the Value of y will be found $= \frac{1}{2}$.

Since (as we have now prov'd) a will be conjoin'd with B in two Hours after it was conjoin'd with it before, it is manifeſt that they will be conjoin'd in every Multiple of 2 Hours, as in 2, 4, 6, 8, 10, &c. Hours. In like manner it will appear that A will be conjoin'd with C in every Multiple $\frac{3}{4}$ Hours as $\frac{3}{4}$, $\frac{6}{4}$, $\frac{9}{4}$, $\frac{12}{4}$, $\frac{15}{4}$ &c. Hours, and that A and D will be conjoin'd in every Multiple of $\frac{1}{2}$ Hour as $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, &c. Hours; wherefore the leaſt Multiple of (or the leaſt Number that can be Divided, without having a Remainder by) 2, $\frac{3}{4}$ and $\frac{1}{2}$ is the Number of Hours wherein the Conjunction of A with B, with C and with D will be accompliſh'd; and of conſequence A, B, C and D will then be on the ſame Point.

Now in order to find the leaſt Number that can be Divided by 2, $\frac{3}{4}$ and $\frac{1}{2}$ without leaving a Remainder, I proceed as my Rule directs thus,

The two firſt taken Numbers, ſuppoſe 2 and $\frac{3}{4}$, being reduc'd to a common Denominator, are $= \frac{8}{4}$ and $\frac{3}{4}$ reſpectively, and the leaſt Number that can be Divided by $\frac{8}{4}$ and $\frac{3}{4}$ without leaving a Remainder, will be found (by the ſaid Rule) $= \frac{8 \times 3}{4 \times 1} = \frac{24}{4}$

And the leaſt Number that can be Divided by $\frac{24}{4}$ and $\frac{1}{2}$ ($\frac{2}{4}$) will be alſo found, by our ſaid Rule (or *Lemma*), $= \frac{24 \times 2}{4 \times 2} = \frac{24}{4} = 6$; wherefore 6 is the Number of Hours wherein A, B, C and D, after their being mov'd from the ſame Point, will be conjoin'd; that is to ſay 6 Hours is the Anſwer required.

Note, *What is here ſaid of Numbers may be applyed to Quantities, as per* Example.

Queſtion 12.

A Clock hath three Indices, A, B, and C. A is carried $\frac{b}{p}$ ($\frac{3}{2}$) times round the whole Circumference of the Clock in one Hour. B is carried the ſame way $\frac{c}{q}$ ($\frac{11}{5}$) times round in one Hour. And C is carried the ſame way $\frac{d}{r}$ ($\frac{17}{7}$) times round the whole Circumference of the ſaid Clock in one Hour. The Queſtion is, ſuppoſing them to be mov'd at once from the ſame Point of the ſaid Circumference, and alſo ſuppoſing $\frac{b}{p} \sqsupset \frac{c}{q}$ alſo $\frac{c}{q} \sqsupset \frac{d}{r}$, in how many Hours from the ſaid Moving will they all three be conjoin'd again.

Suppoſe A to be conjoin'd with $\left\{\begin{matrix} B \\ C \end{matrix}\right\}$ In $\left\{\begin{matrix} a \\ e \end{matrix}\right\}$ Hours.

Since A's Motion is the ſloweſt, it is evident that either of the reſt, towit B or C will be carried 1 Circumference more than A at the time that it will overtake (or be conjoin'd with) A; wherefore $\frac{ab}{p} + 1$ Circumferences $= \frac{ac}{q}$ Circumferences.

Therefore $abq + pq = acp$
And by Tranſpoſition $pq = acp - abq$

And by Diviſion $\frac{pq}{cp - bq} =$ [illegible]

And in like Manner e will be found $= \frac{pr}{pd - rb}$

Wherefore A will be conjoin'd with B in every Multiple of $\frac{pq}{cp - bq}$ Hours; as $\frac{pq}{cp - bq}$, $\frac{2pq}{cp - pq}$, $\frac{3pq}{cp - pq}$, &c. Hours, and A will be conjoin'd with C in every Multiple of $\frac{pr}{pd - rb}$ Hours; as $\frac{pr}{pd - rb}$, $\frac{2pr}{pd - rb}$, $\frac{3pr}{pd - rb}$, &c. Hours; and therefore A will be conjoin'd with B, as alſo with C. and conſequently B will be conjoin'd with C, in every Quantity that can be Divided by $\frac{pq}{cp - bq}$ and $\frac{pr}{pd - rb}$ without leaving a Remainder. Conſequently the leaſt Quantity that can be Divided by $\frac{pq}{cp - bq}$ and $\frac{pr}{pd - rb}$ without leaving a Remainder, is $=$ the Number of Hours wherein the Conjunction of A, B and C will be accompliſh'd, or the Anſwer required.

Now $\frac{pq}{cp - bq}$ and $\frac{pr}{pd - rb}$, being reduc'd to a common Denominator, are equal to $\frac{p^2qd - pqrb}{p^2cd - pcrb - qbpd + qrb^2}$ and $\frac{p^2cr - prqb}{p^2cd - pcrb - qbpd + qrb^2}$: and the leaſt Quantity that can be Divided by them, without having any Remainder will be found, by our *Lemma* (putting $w =$ the greateſt common Meaſure of $pqd - qrb$ and $pcr - brq$) to be

$$\frac{p^3qdcr - p^2q^2drb - p^2qr^2bc + pq^2r^2b^2}{\overline{p^2cd - pcrb - qbpd + qrb^2} \times w} = \frac{pqr}{w} \quad \left(= \frac{2 \times 5 \times 7}{1} = 70\right)$$

Hours Anſwer.

Sect. 2. Of Unlimited Queſtions.

I will now proceed to give an Example or two of the Method uſed in arguing about unlimited Queſtions, *viz.* ſuch Queſtions as admit of various Anſwers, ſuch as thoſe in Alligation Alternate.

Example 1. *Queſtion* 13.

A Tobacconiſt hath three ſorts of Tobacco, *viz.* one of 2 *s.* 8 *d.* the Pound, another of 20 *d.* the Pound, and a third Sort of 16 *d.*

the Pound: Of theſe he would make a Mixture to contain 56 Pound that may be Sold for 22*d.* the Pound. How much of each Sort may he take?

Let a = the Quantity of that Worth 2*s.* and 8*d.* or 32 Pence the Pound, e = that of 20 Pence the Pound, and y = that of 16 Pence the Pound;

Then $a + e + y = 56$
And $32a + 20e + 16y = 56 \times 22 = 1232$

viz. each Quantity Multiplied into its own Price, equals their Sum Multiplied into the Mean Price.

This Queſtion being thus Stated, it appears (by Part VII. Rule 1) that it is capable of innumerable Anſwers; becauſe for any one of theſe three Letters $a.$ $e.$ $y.$ there may be taken any Number at Pleaſure, provided it be Leſs than 56. But altho' that may be truly done, yet there are ſeveral ways of Arguing about theſe ſort of Queſtions, which will Limit or Bound them to all their proper or poſſible Anſwers in whole Numbers. Thus,

Let	1	$a + e + y = 56$	As above.
And	2	$32a + 20e + 16y = 1232$	
$1 - a$	3	$e + y = 56 - a$	
$2 - 32a$	4	$20e + 16y = 1232 - 32a$	
3×16	5	$16e + 16y = 896 - 16a$	
$4 - 5$	6	$4e = 336 - 16a$	
$6 \div 4$	7	$e = 84 - 4a.$	Hence $a \sqsupset \frac{84}{4}(21)$ *
$3 - 7$	8	$y = 3a - 28.$	Hence $a \sqsubset \frac{28}{3}(9\frac{1}{3})$ *

* becauſe (by Fra an Diviſ.) $a + \frac{e}{4}$; $= \frac{84}{4}$; and $a - \frac{1}{3}$ is $= \frac{28}{3}$.

From the two laſt Steps it appears, that the Quantity ſignified by a, ought to be Leſs than 21, and Greater than $9\frac{1}{3}$; that is, any Number betwixt $9\frac{1}{3}$ and 21, may be taken for the Value of a. Conſequently there may be Eleven Anſwers to this Queſtion in whole Numbers.

Suppoſe $a = 10$. Then $e = 84 - 40 = 44$ *per* 7*th* Step. and $y = 30 - 28 = 2$ *per* 8*th* Step.

Again, if $a = 11$ Then $e = 84 - 44 = 40$ *per* 7*th* Step. And $y = 33 - 28 = 5$ *per* 8*th* Step. And ſo on for the Reſt, which will be as in the following Table.

a	e	y	a	e	y	a	e	y
10	44	2	14	28	14	18	12	26
11	40	5	15	24	17	19	8	29
12	36	8	16	20	20	20	4	32
13	32	11	17	16	23			

Thus it will be eaſy to find out and Collect all the Limited Anſwers to any Queſtion (of this kind) wherein there are only three Quantities propoſ'd to be Mix'd: But when there are more than Three, then the Work requires a little more Trouble; becauſe the ſingle Limits of all the Quantities above Two, muſt be found. That is, if there are four Quantities concern'd in the Queſtion, the Limits of Two of them muſt be found; if five Quantities are concern'd, then the Limits of Three of them muſt be found, &c. As in the following Queſtion.

Queſtion 14.

Suppoſe it were required to mix Four Sorts of Wines together; *viz.* one Sort worth 7*s.* 4*d.* the Gallon; another Sort worth 4*s.* 7*d.* the Gallon; a third Sort worth 3*s.* 8*d.* the Gallon; and a fourth Sort worth 2*s.* 9*d.* the Gallon; How many of each Sort may be taken to make a Mixture of 63 Gallons, ſo as that the whole Quantity may be Sold for 5*s.* 6*d.* the Gallon; without Loſs, &c.

First let all theſe ſeveral Rates, and the Mean Rate, be reduc'd to one Denomination, *viz.* into Pence.

Viz. 7*s.* 4*d.* = 88*d*; 4*s.* 7*d.* = 55*d*; 3*s.* 8*d.* = 44*d*; 2*s.* 9*d.* = 33*d*; and 5*s.* 6*d.* = 66. Then put a = the Quantity of that Worth 88*d.* the Gallon; e = that of 55*d.* the Gallon; y = that of 44*d.* the Gallon; and u = that of 33*d.* the Gallon. Then

	1	$a+e+y+u=63$ By the Queſtion;
and	2	$88a+55e+44y+33u=63\times66=4158$
$1-a$	3	$e+y+u=63-a$
$2-88a$	4	$55e+44y+33u=4158-88a$
3×33	5	$33e+33y+33u=2079-33a$
$4-5$	6	$22e+11y=2079-55a$
$6\div11$	7	$2e+y=189-5a$. Hence a is $\sqsupset\frac{189}{5}$ $(37\frac{4}{5})$
3×55	8	$55e+55y+55u=3465-55a$
$8-4$	9	$11y+22u=33a-693$.
$9\div11$	10	$y+2u=3a-63$ Hence $a\sqsubset\frac{63}{3}$ (21).

From the 7*th* and 10*th* Steps, it appears, that the Quantity of that ſort of Wine denoted by a muſt be leſs than $37\frac{4}{5}$ Gallons, and greater than 21 Gallons: that is it may be a = any Number of Gallons betwixt 21 and $37\frac{4}{5}$; wherefore it follows that there may be Collected 16 Anſwers to this Queſtion from the Limits of a only.

Next to find the Limits of e, y and u.

Suppoſe

Suppoſe	11	$a = 22$, then will $5a = 110$, and $3a = 66$
But	12	$2e + y = 189 - 5a = 79$ *Per* 7*th* Step.
$12 - 2e$	13	$y = 79 - 2e$. Hence $e \sqsupset \frac{79}{2} \, (39\frac{1}{2})$
Again	14	$e + y + u = 63 - a = 41$ *Per* 3*d* Step.
$14 - e$	15	$y + u = 41 - e$
$15 - 13$	16	$u = + e - 38$ Hence $e \sqsubset 38$.

From the 13*th* and 16*th* Steps, it appears, that if $a = 22$, then $e = 39$, $y = 79 - 2e = 1$, and $u = e - 38 = 1$.

Again,

Suppoſe	17	$a = 23$, then $5a = 115$, and $3a = 69$
But	18	$2e + y = 189 - 5a = 74$ *Per* 7*th* Step.
$18 - 2e$	19	$y = 74 - 2e$ Hence $e \sqsupset \frac{74}{2} \, (37)$
Again	20	$e + y + u = 63 - a = 40$ *Per* 3*d* Step.
$20 - e$	21	$y + u = 40 - e$
$21 - 19$	22	$u = e - 34$. Hence $e \sqsubset 34$.

From the 19*th* and 22*d* Steps, it appears, that if $a = 23$, then e may be either 35 or 36.

Once more for a further Illuſtration.

Let	23	$a = 24$; then $5a = 120$, and $3a = 72$
But	24	$2e + y = 189 - 5a = 69$ *Per* 7*th* Step.
$24 - 2e$	25	$y = 69 - 2e$. Hence $e \sqsupset \frac{69}{2} \, (34\frac{1}{2})$
Again	26	$e + y + u = 63 - a = 39$ *Per* 3*d* Step.
$26 - e$	27	$y + u = 39 - e$
$27 - 25$	28	$u = e - 30$. Hence $e \sqsubset 30$.

From hence it appears, that if $a = 24$, then e may be either 31, 32, 33, or 34, *viz.* it may be any Number betwixt 30 and $34\frac{1}{2}$, by the 25*th* and 28*th* Steps. From whence the Values of y and u may be eaſily found;

$$\text{That is, if } \left\{ \begin{array}{l} e = 31 \\ e = 32 \\ e = 33 \\ e = 34 \end{array} \right\} \text{ Then } \left\{ \begin{array}{l} y = 7 \\ y = 5 \\ y = 3 \\ y = 1 \end{array} \right\} \text{ And } \left\{ \begin{array}{l} u = 1 \\ u = 2 \\ u = 3 \\ u = 4 \end{array} \right\}$$

Proceeding in this manner with all the other Values of a, there may be above 120 Anſwers found to this Queſtion in whole Numbers, and, if you pleaſe to put $a =$ to Fractions, there may be found an innumerable Set of Anſwers.

Theſe two Examples being well Underſtood (eſpecially if the laſt be throughly purſued) may ſuffice to ſhew the Method of Limiting Anſwers, to all ſorts of Queſtions of this kind.

Sect. 3. Queſtions producing Simple Quadratick &c. Æquations.

Queſtion 15.

A certain Footman A departed from *London* towards *Lincoln*, and at the ſame time another Footman B departed from *Lincoln* towards *London*, each keeping the ſame Road. When they met A ſaid to B I find I have travell'd b (20) Miles more than you, and have gone as many Miles in c ($6\frac{2}{3}$) Days as you have gone Miles in all hitherto. But ſaid B at the end of d (15) Days Hence I ſhall be at *London*, if I Travel ſtill after the ſame Rate. *Quære* the Diſtance of theſe two Cities from one another, and how many Miles each Footman had travell'd when they met,

1. Suppoſe the Number of Miles that A went daily, to be $= a$; the Number of Days that he was on the Road to the time that he met B, $= z$; then the Number of Miles that he went to the ſaid Time that he met B will be $= za$.

2. Suppoſe the Number of Miles that B went every Day $= e$, then the Number of Days that B was on the Road to the time he met A is (ſince B parted from *Lincoln* at the ſame time that A Parted from *London*) $= z$; and the Number of Miles that he went to the Time that he met A, will be $= ze$.

	3	$za = ze + b$	*Per* Queſtion.
	4	$ca = ze$	
	5	$de = za$	
$4 \div c$	6	$a = \frac{ze}{c}$	
$5 \div z$	7	$a = \frac{de}{z}$	
$6 = 7$	8	$\frac{ze}{c} = \frac{de}{z}$	
8 reduc'd	9	$z = \sqrt{cd}\,(\overline{6\frac{2}{3} \times 15}\,|^{\frac{1}{2}} = \sqrt{100} = 10$ Days)	
$3 \div z$	10	$a = \frac{ze + b}{z}$	
From 7 and 10	11	$\frac{de}{z} = \frac{ze + b}{z}$	
$11 \times z$	12	$de = ze + b$	

$b = 20$
$c = \frac{20}{3}$
$d = 15$

$de =$

From 9 and 12	13	$de = e\sqrt{cd} + b$, By Subſtit.
$13 - e\sqrt{cd}$	14	$de - e\sqrt{cd} = b$
$14 \div \overline{d - \sqrt{cd}}$	15	$e = \frac{b}{d - \sqrt{cd}} \left(= \frac{20}{15 - \sqrt{100}} = 4\right.$ Miles.)
And from 6, 7, or 10, 9, & 15 Steps.	16	$a = \frac{b}{\sqrt{cd} - c} \left(= \frac{20}{\sqrt{100} - 6\frac{2}{3}} = 6\right.$ Miles.)
From 9, 15 and 16.	17	$za + ze = \frac{b\sqrt{cd}}{\sqrt{cd} - c} + \frac{b\sqrt{cd}}{d - \sqrt{cd}}$

$$\left(= \frac{20\sqrt{100}}{\sqrt{100} - 6\frac{2}{3}} + \frac{20\sqrt{100}}{15 - \sqrt{100}} = \frac{200}{3\frac{1}{3}} + \frac{200}{5} = 60 + 40\right.$$

Miles) = the Number of Miles that A and B went when they met; that is = the Diſtance in Miles from *London* to *Lincoln*.

Queſtion 16.

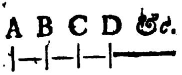
There is a Wall containing b (18225) Cubical Feet, the Height is c (5) times the Breadth: the Length is d (8) times the Height. I demand the Breadth of the ſaid Wall.

	1	Suppoſe the Number of Feet in the Breadth of the Wall to be $= a$.
	2	Then the Number of Feet in its Height is $= ca$.
	3	And the Number of Feet in its Length is $= dca$
	4	Now by the Queſtion $a \times ca \times dca = b = dc^2a^3$
$4 \div dc^2$	5	$\frac{b}{dcc} = aaa$
5 ɯ 3	6	$\sqrt[3]{\frac{b}{dcc}} = a \left(= \sqrt[3]{\frac{18225}{8 \times 5 \times 5}} = \sqrt[3]{\frac{18225}{200}} = \overline{91.125}\vert^{\frac{1}{3}} = 4.5\right.$ Feet).

CHAP. II.

Queſtions producing Affected Quadratick Æquations.

Lemma.



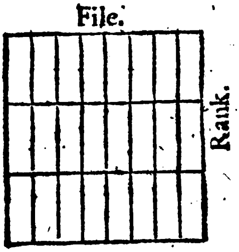
IF along with the one thing A, in any right Line be plac'd any Number of things B, C, D, &c: at any Diſtance, or Diſtances from one another; I ſay the Number of the ſaid things A, B, C, D, &c. is by one more than the Number of the Intervals A B, B C, C D, &c.

Demonſtration.

If in the ſaid right Line, along with, and at any Diſtances from, the one thing A, you place n things B, C, D, &c. (n being equal to any whole Number whatſoever) it is Evident that you place n Intervals A B, B C, C D, &c. Conſequently the Number of things is $n + 1$, and the Number of Intervals is n. Q. E. D.

Queſtion 1.

b (36) Soldiers are to be plac'd in a Square Battle in ſuch manner, that the Soldiers in Rank may have c (8) Foot Diſtance between every two, and the Soldiers in File may have d (3) Foot Diſtance between each two. How many Soldiers muſt be plac'd in Rank, and how many in File?



1. Suppoſe the Number of Feet in the Side of the Square $= a$.

Then (by the Nature of the Queſtion, and by the foregoing *Lemma*)

2. The Number in Rank is $= \frac{a}{c} + 1 = \frac{a+c}{c}$.

3. And the Number in File is $= \frac{a}{d} + 1 = \frac{a+d}{d}$.

4. Therefore $\frac{aa \begin{matrix}+c\\+d\end{matrix} a + cd}{cd} = b$.

$4 \times cd$	5	$a^2 \begin{matrix}+c\\+d\end{matrix} a + cd = bcd$
$5 - cd$	6	$a^2 + \overline{c+d} \times a = bcd - cd$
	7	Suppose $c + d = g$; then the foregoing Æquation will become $a^2 + ga = bcd - cd$
Compleat. □	8	$a^2 + ga + \frac{1}{4}gg = bcd - cd + \frac{1}{4}gg$
8 ɯ 2	9	$a + \frac{1}{2}g = \sqrt{\overline{bcd - cd + \frac{1}{4}gg}}$
$9 - \frac{1}{2}g$	10	$a = \sqrt{\overline{bcd - cd + \frac{1}{4}gg}} - \frac{1}{2}g$ (= 24)
	11	Consequently the Number in Rank is = $\frac{\sqrt{\overline{bcd - cd + \frac{1}{4}gg}} - \frac{1}{2}g}{c} + 1$ (= 4)
	12	And the Number in File is = $\frac{\sqrt{\overline{bcd - cd + \frac{1}{4}gg}} - \frac{1}{2}g}{d} + 1$ (= 9)

Question 2.

The Sum (b) of the Squares of two Numbers being given, as alſo (c) the double Product of the Multiplication of the ſame two Numbers; to find the Numbers ſeverally.

Solution.

	1	Suppoſe the greater Number $= a$
	2	And the leſſer Number $= e$
	3	Then $aa + ee = b$ } *Per* Queſtion. =100
	4	And $2ae = c$ } *Per* Queſtion. =96
$4 \div 2a$	5	$e = \frac{c}{2a}$
5 Ⓖ 2	6	$e^2 = \frac{cc}{4aa}$
From 3 and 6	7	$a^2 + \frac{cc}{4aa} = b$
$7 \times 4a^2$	8	$4a^4 + cc = 4ba^2$

8 ÷ 4

$8 \div 4$	9	$a^4 + \frac{1}{4}cc = ba^2$
$9 - \frac{1}{4}c^2$	10	$a^4 = ba^2 - \frac{cc}{4}$
$10 - ba^2$	11	$a^4 - ba^2 = -\frac{cc}{4}$
Compleat. □	12	$a^4 - ba^2 + \frac{1}{4}bb = \frac{1}{4}bb - \frac{1}{4}cc$
12 ω 2	13	$a^2 - \frac{1}{2}b = \pm\overline{\frac{b^2 - c^2}{4}}\Big\|^{\frac{1}{2}}$
$13 + \frac{b}{2}$	14	$a^2 = \pm\frac{\sqrt{bb - cc} + b}{2}$ *
14 ω 2	15	$a = \sqrt{\frac{\pm\sqrt{bb - cc} + b}{2}}$
$3 - e^2$	16	$a^2 = b - e^2$
$14 = 16$	17	$\pm\frac{\sqrt{bb - cc} + b}{2} = b - e^2$
And by Tranſ.	18	$e^2 = \frac{b \mp \sqrt{bb - cc}}{2}$
18 ω 2	19	$e = \sqrt{\frac{b \mp \sqrt{bb - cc}}{2}}$

* becauſe $\sqrt{\frac{b^2 - c^2}{4}}$ is $= \frac{\sqrt{b^2 - c^2}}{\sqrt{4}} = \frac{\sqrt{b^2 - c^2}}{2}$

Scholium.

From the 15*th* and 19*th* Steps ariſes this *Canon* for Extracting the Square-Root of Binomial or Reſidual Numbers.

Canon.

From the Square of the given Sum of the Squares (or from the greater part of the given Binomial or Reſidual) Subſtract the Square of the double Product given, (or Subſtract the Square of the leſſer Part) then Add and Subſtract the Square-Root of the Remainder to and from the given Sum of the Squares (or to and from the ſaid greater Part). Laſtly Connect the Square-Roots the half of that Sum and Remainder by the Sign +, if a Binomial be propos'd; but by —, if a Reſidual, ſo you'll have the deſired Square-Root.

Example.

Let it be required to Extract the Square Root of $27 + \sqrt{704}$.

Operation.

Operation.

From the Square of the greater Part 27, *viz.*	729
Subſtract the Square of the leſſer Part $\sqrt{704}$, *viz.*	704
And the Remainder is	25
The Square-Root of which Remainder is	5
Which added to the greateſt Part 27 is	32
The half of which Sum is	16
The Square-Root of which is the greater Number	$= 4$
The ſaid Square-Root 5 Subſtracted from the greater Part 27 leaves	22
The half of which is	11
And the Square-Root thereof is the leſſer Number	$= \sqrt{11}$

I ſay the Root ſought ſhall be the two Parts ſo found Connected by $+$; that is $4 + \sqrt{11}$.

But if $-$ inſtead of $+$ be prefixt to the leſſer Part, then the Square Root ſought (that is the Square-Root of $27 - \sqrt{704}$) will be $4 - \sqrt{11}$.

Lemma.

If two Men A and B make a Joint-Stock, ſo as that their Sums of Money are not in Company for equal times: I ſay that as A's Stock Multiplyed by the time of its being out, is to his Gain, ſo is B's Stock Multiplyed by the time of its being out, to his Gain.

Demonſtration.

Suppoſe the Product of A's Stock Multiplyed by his time $= b$.
The Product of B's Stock Multiplyed by his time $= c$.
And their Gain $= n$.
Then (by the Rule of Fellowſhip with time).

A's Gain is $= \frac{bn}{b+c}$.

And B's Gain is $= \frac{nc}{b+c}$.

And $b \,..\, \frac{bn}{b+c} :: c \,..\, \frac{cn}{b+c}$, as is evident by Æquating the Product of the Extreams to that of the Means. *Q. E. D.*

Queſtion 3.

Two Men, *viz.* D and E made a Stock of b (165) Pounds. D's Money was in Company for c (12) Months, and E's Money was in for d (8) Months. When they ſhar'd Stock and Gain, D receiv'd f (67) Pounds, and E, g (126) Pounds. I demand each Man's Stock?

1. Suppoſe D's Stock $= a$; then the Product of his Stock and Time is $= ca$: and his Gain $= f - a$

2. Then E's Stock $= b - a$ and the Product of his Stock and Time $= db - da$: and his Gain $= g - b + a$

Then by the preceeding *Lemma.*	3	$ca \,..\, db - da :: f - a \,..\, g - b + a$
Æquation	4	$cga - cba + caa = fdb - fda - dba + daa$
Tranſpoſition	5	$caa - daa + fda + dba + cga - cba = fdb$
$5 \div \overline{c-d}$	6	$a^2 + \frac{fd + db + cg - cb}{c-d} \times a = \frac{fdb}{c-d}$
		If you put b for, and $= \frac{fd + db + cg - cb}{c-d}$
	7	Then is $aa + ba = \frac{fdb}{c-d}$
Compleat. □	8	$aa + ba + \frac{1}{4}bb = \frac{fdb}{c-d} + \frac{1}{4}bb$
8 ɯ 2	9	$a + \frac{b}{2} = \sqrt{\frac{fdb}{c-d} + \frac{1}{4}bb}$
$9 - \frac{1}{2}b$	10	$a = \sqrt{\frac{fdb}{c-d} + \frac{1}{4}bb} - \frac{1}{2}b$ (= 54 l.)

N. B. *Part XII begins Page* 177, *and Signature* a.

PART XII.

Of ſeveral Methods of Solving High Adfected Æquations.

CHAP. I.

Some Preparations ſometimes convenient for the Solution of High ADFECTED ÆQUATIONS.

PROP. I.

TO Increaſe or Diminiſh the Value of the unknown Roots of an Æquation by any given Quantity.

RULE.

Inſtead of the unknown Root, ſubſtitute another unknown Root, Leſs or Greater than it, by the Quantity given. Thus,

Example I. To Augment the Roots of this Æquation $aa + ba - dd = 0$, by the Quantity c.

Put $a = e - c$, then $aa + ba - dd = 0$ is $=$ $e^2 - 2ec + cc + be - bc - dd = 0$.

Whoſe Roots (e) exceed them of the former, (a) by the Quantity c; that is, $e = c + a$.

Example II. To diminiſh the Roots of this Æquation $a^2 + ba - dd = 0$, by the Quantity c.

Put $e + c = a$, and then you'll have this Æquation.

$$ee + 2ec + cc + be + bc - dd = 0.$$

Whoſe Roots are leſs than thoſe of the Æquation propos'd, by the Quantity c.

SCHOLIUM I.

By this manner of Subſtitution, any Term, except the firſt of any given Æquation, may be deſtroyed.

So if the Æquation given be $x^4 - bx^3 + cx^2 - dx + f = 0$.

Put $y + n = x$.

Then

$$\begin{array}{rrrrrcl} y^4 & +4ny^3 & +6n^2y^2 & +4n^3y & +n^4 & = & xxxx \\ & -by^3 & -3bny^2 & -3bn^2y & -bn^3 & = & -bxxx \\ & & + \; cy^2 & +2cny & +cn^2 & = & +cxx \\ & & & -dy & -dn & = & -dx \\ & & & & +f & = & +f \end{array}$$

Now 'tis plain, that any Term, except the firſt, may be taken away from this Æquation, becauſe n was taken at pleaſure. *Viz.*

Putting $4n - b = 0$, the ſecond Term muſt vaniſh; and putting $6n^2 - 3bn + c = 0$, the third Term will vaniſh.

In like manner any other Term in this, or any other Æquation may be deſtroyed, except the firſt.

Therefore in taking away the ſecond Term of any Æquation, let the Index of the higheſt Power of the ſought Root x be $= m$; and ſuppoſing the Coefficient of the ſecond Term be $= \pm a$. Subſtitute $y \mp \frac{a}{m}$ in the room of x, every where in the given Æquation, and you'll have another, in which y will be the Root ſought, and which will want the ſecond Term. Then after you have found the Value of y, you may find that of x, for $y = x \pm \frac{a}{m}$.

SCHOLIUM II.

By this way of Subſtitution you may add to any Æquation any Term that it wants.

PREPA-

As ſuppoſing the Æquation to be $x^4 + c^3x - d^4 = 0$. Put $y - c = x$.

Then,

$$\begin{aligned} y^4 - 4cy^3 + 6c^2y^2 - 4c^3y + c^4 &= x^4 \\ + c^3y - c^4 &= c^3x \\ - d^4 &= -d^4 \\ \hline y^4 - 4cy^3 + 6c^2y^2 - 3c^3y - d^4 &= 0. \end{aligned}$$

PROP. II.

To Multiply or Divide the Roots of an Æquation by any given Quantity: Suppoſe q.

RULE.

Inſtead of the unknown Root ſubſtitute $\frac{1}{q}$ part, or q times, another unknown Root: Thus,

Example I. To multiply the Roots of this Æquation $x^3 + bx^2 - c^2x = g^3$ by q.

Suppoſe $\frac{1}{q} \times z = x$; then the former Æquation will become,

$$\frac{z^3}{q^3} + \frac{bz^2}{q^2} - \frac{c^2z}{q} = g^3.$$

And Multiplying each Part by q^3 we have $z^3 + qbz^2 + q^2c^2z = q^3g^3$, an Æquation wherein z is $= qx$ in the former.

The Reaſon is obvious, for if $\frac{1}{q}z$ be $= x$, then $z = qx$.

Example II. To Divide by 3 the Roots of this Æquation, $x^3 - 54c^2x - 216d^3 = 0$.

Suppoſe $3 \times z = x$, then the former Æquation will become $27z^3 - 162c^2z - 216d^3 = 0$.

And by Dividing each Part by 27, we have $z^3 - 6c^2z - 8d^3 = 0$; an Æquation wherein z is $= \frac{x}{3}$ in the former

COROLLARY.

Hence an Æquation may be clear'd from Fractions, without increaſing the Coefficient of the higheſt Term; thus, if $x^3 + \frac{1}{3}x - \frac{40}{7} = 0$; that is, (firſt reducing $\frac{1}{3}x - \frac{40}{7}$ to a common Denominator) if $x^3 + \frac{7}{21}x - \frac{120}{21} = 0$.

I put $q = 21$, and $\frac{z}{q} = x = \frac{z}{21}$, then the former Æquation becomes $= \frac{z^3}{9261} + \frac{7z}{441} - \frac{120}{21} = 0$; and by Multiplying each Part by 9261, we have $z^3 + 147z - 52920 = 0$.

See the Uſe of this Corollary *in Page*

CHAP. II.

The Solution of CUBICK ÆQUATIONS, by *Cardan*'s Method.

WHen any Cubick Æquation having the ſecond Term, be propoſ'd to be reſolv'd, by *Cardan*'s Method, you muſt firſt deſtroy its ſecond Term (by the laſt *Chap. Propoſition I. Scholium I.*) and then you'll reduce it into one of theſe four following Forms or Caſes, *Viz.*

$$\begin{array}{c|l}
1 & a^3 * + pa = q \\
2 & a^3 * - pa = q \\
3 & a^3 * - pa = -q \\
4 & a^3 * + pa = -q
\end{array}$$

In either of thoſe Æquations the Quantity ſought (a) is inſerted only in two different Terms, in which its Indices are triple to one another: And in order to ſolve all Æquations thus qualified at once, let us ſuppoſe the reduc'd Æquation to be equal to $x^n + bx^{\frac{n}{3}} = c$, in which the Value of b and c may be either equal to any given Quantity whatſoever, Affirmative or Negative; and that of n is indetermin'd, but is in all original Cubick Æquations $= 3$; and generally in all other Cubicks, equal to ſome Multiple of 3, and x^n is underſtood to have the Sign + prefix'd to it.

SOLUTION.

If	1	$x^n + bx^{\frac{n}{3}} = c$
Suppoſe	2	$z = \overline{x}\vert^{\frac{n}{3}}$
From 1 and 2	3	$z^3 + bz = c$
Suppoſe	4	$e + y = z$
Then	5	$e^3 + 3eey + 3eyy + y^3 + b \times \overline{e+y}$ $(= z^3 + bz) = c$.
Now ſuppoſe	6	$b = -3ey$
From 5 and 6	7	$e^3 + 3eey + 3eyy + y^3 - 3eey - 3eyy = c = e^3 + y^3$
$6 \div -3e$	8	$-\frac{b}{3e} = y$
8 ⊙ 3	9	$-\frac{bbb}{27eee} = y^3$
From 7 and 9	10	$c = e^3 - \frac{b^3}{27e^3}$
10 × e^3	11	$ce^3 = e^6 - \frac{b^3}{27}$
$11 + \frac{b^3}{27} - ce^3$	12	$\frac{b^3}{27} = e^6 - ce^3$
Comp. □	13	$\frac{cc}{4} + \frac{b^3}{27} = e^6 - ce^3 + \frac{cc}{4}$
13 ɯ 2	14	$\sqrt{\frac{cc}{4} + \frac{b^3}{27}} = e^3 - \frac{c}{2}$

$14+\frac{c}{2}$	15	$\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}=e^3$
15 ω 3	16	$\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{bbb}{27}}}=e$
From 16 and 8	17	$\frac{b}{3\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{bbb}{27}}}}=y$
From 16, 17 & 4	18	$\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{bbb}{27}}}-\frac{\frac{1}{3}b}{\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{bbb}{27}}}}=z=x^{\frac{n}{3}}$
18 ω $\frac{n}{3}$	19	$\sqrt[\frac{n}{3}]{\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}}-\frac{\frac{1}{3}b}{\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{bbb}{27}}}}}=x.$

SCHOLIUM.

As the Cube Root of any Quantity k, is not only $\sqrt[3]{k}$, or $1\times\sqrt[3]{k}$, but alſo $\overline{-\frac{1}{2}+\sqrt{-\frac{3}{4}}}\times\sqrt[3]{k}$, and $\overline{-\frac{1}{2}-\sqrt{-\frac{3}{4}}}\times\sqrt[3]{k}$:* So the Value of z in the Æquation $z^3+bz=c$, is not only $1\times\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}}-\frac{\frac{1}{3}b}{1\times\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{bbb}{27}}}}$ (as in the 18th Step next foregoing); but alſo $\overline{-\frac{1}{2}+\sqrt{-\frac{3}{4}}}\times\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}}-\frac{\frac{1}{3}b}{\overline{-\frac{1}{2}+\sqrt{-\frac{3}{4}}}\times\sqrt[3]{\frac{c}{2}+\sqrt{\frac{c}{4}+\frac{bbb}{27}}}}$

+ &c

* because the Cube of $\overline{-\frac{1}{2}+\sqrt{-\frac{3}{4}}}\times\sqrt[3]{k}$ will be found and ~~to be~~ $\frac{1}{4}\sqrt[3]{k^3}+\frac{3}{4}\sqrt[3]{k^3}=\sqrt[3]{k^3}=k$; so is k also equal to

$\overline{-\frac{1}{2}-\sqrt{-\frac{3}{4}}}^3\times\sqrt[3]{k}$ as upon Trial you will find.

and $\overline{-\frac{1}{2}-\sqrt{-\frac{3}{4}}}\times\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}}-$

$$\frac{\frac{1}{3}b}{\overline{-\frac{1}{2}-\sqrt{-\frac{3}{4}}}\times\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{bbb}{27}}}}$$: For all three are but the ſame Values of z differently expreſs'd; the firſt of them being the Simpleſt, is ~~therefore the Beſt~~.

That theſe are the three Values of z, in the Æquation $z^3+bz=c$, I demonſtrate thus:

Suppoſe $u=-\frac{1}{2}+\sqrt{-\frac{3}{4}}$, $v=-\frac{1}{2}-\sqrt{-\frac{3}{4}}$, and $w=\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}}$; then the foregoing Values of z will become $w-\frac{\frac{1}{3}b}{w}$, $uw-\frac{\frac{1}{3}b}{uw}$, and $vw-\frac{\frac{1}{3}b}{vw}$, reſpectively; wherefore $z-w+\frac{\frac{1}{3}b}{w}=0$, $z-uw+\frac{\frac{1}{3}b}{uw}=0$, and $z-vw+\frac{\frac{1}{3}b}{vw}=0$: and the Product of theſe two laſt Æquations will be found when reduc'd =

$$\begin{array}{lll} z^2+w & & +w^2 \\ -\frac{\frac{1}{3}b}{w} & z & +\frac{1}{3}b \quad =0, \\ & & +\frac{\frac{1}{9}bb}{ww} \end{array}$$

($u+v$ being $=-1$, $vu=1$, and $u^2+v^2=-1$).

And this Product Multiplied by $z-w+\frac{\frac{1}{3}b}{w}=0$, gives $z^3+bz-w^3+\frac{\frac{1}{27}b^3}{w^3}=0$; that is, $z^3+bz-c=0$; for $-w^3+\frac{\frac{1}{27}b^3}{w^3}$, may be eaſily prov'd to be $=c$, ∧ from 7, 15, and 17 Steps. *Q. E. D.*

COROL.

COROLLARY.

From the 18th Step next foregoing you may deduce particular Theorems (but not the ſame with *Cardan's*) for Solving each of the four preceding Caſes.

So 1. If $a^3+pa=q$; then (b in the Canon being $=$ p in this Æquation, and c in the former $=q$ in the latter)

$$a=\sqrt[3]{\frac{q}{2}+\sqrt{\frac{qq}{4}+\frac{p^3}{27}}}-\frac{\frac{1}{3}p}{\sqrt[3]{\frac{q}{2}+\sqrt{\frac{qq}{4}+\frac{ppp}{27}}}}.$$

Alſo 2. If $a^3-pa=q$; then ($-p$ being $=b$ and $q=c$)

$$a=\sqrt[3]{\frac{q}{2}+\sqrt{\frac{qq}{4}-\frac{p^3}{27}}}+\frac{\frac{1}{3}p}{\sqrt[3]{\frac{q}{2}+\sqrt{\frac{qq}{4}-\frac{ppp}{27}}}}.$$

Again 3. If $a^3-pa=-q$; then,

$$a=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{qq}{4}-\frac{p^3}{27}}}+\frac{\frac{1}{3}p}{\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{qq}{4}-\frac{p^3}{27}}}}.$$

Example I. *Caſe* I.

If i^3+3i^2+[illegible]$i=13$, and one of the Values of i required.

Firſt to deſtroy the ſecond Term, I ſuppoſe $a-1=i$, then the foregoing Æquation will become $a^3+6a=20$; and by the firſt particular *Theorem* $a=\sqrt[3]{10+\sqrt{100+8}}-\frac{2}{\sqrt[3]{10+\sqrt{108}}}=\sqrt[3]{20.392304845413}$, &c. $-\frac{2}{\sqrt[3]{20.392304845413}, \text{ \&c.}}=2.73205$, &c. $-\frac{2}{2.73205, \text{ \&c.}}$ $=2.73205$, &c. $-.73205$, &c. $=2$: But i is (by Suppoſition) $=a-1$; conſequently i is $=1$. *The two other Values of* i *are Imaginary.*

Exam-

Example II. *Caſe* III.

If $a^3 - 21a = -20$; then $a = \sqrt[3]{-10 + \sqrt{100 - 343}} + \frac{7}{\sqrt[3]{-10 + \sqrt{-243}}} = 2 + \sqrt{-3} + \frac{7}{2 + \sqrt{-3}} = 2 + \sqrt{-3} + 2 - \sqrt{-3} = 4$: or $a = \overline{-\frac{1}{2} + \sqrt{-\frac{3}{4}}} \times \overline{2 + \sqrt{-3}} + \frac{7}{\overline{-\frac{1}{2} + \sqrt{-\frac{3}{4}}} \times \overline{2 + \sqrt{-3}}} = -2\frac{1}{2} + \frac{1}{2}\sqrt{-3} - 2\frac{1}{2} - \frac{1}{2}\sqrt{-3} = -5$: or a is $= \overline{-\frac{1}{2} - \sqrt{-\frac{3}{4}}} \times \overline{2 + \sqrt{-3}} + \frac{7}{\overline{-\frac{1}{2} - \sqrt{-\frac{3}{4}}} \times \overline{2 + \sqrt{-3}}} = \frac{1}{2} - \frac{3}{2}\sqrt{-3} + \frac{1}{2} + \frac{3}{2}\sqrt{-3} = 1$.

Tho' the foregoing, and *Cardan's Theorems* are equally true, yet as to Matters of Practice, in as much as Diviſion is eaſier than the Extraction of the Cube Root, the former are preferable, except only when the Value of $\frac{cc}{4} + \frac{b^3}{27}$ is $\sqsupset 0$: In that reſpect they were by ſome deem'd Impracticable and Falſe; but you may ſee by this laſt *Example*, that thoſe are not ſo: 'Tis true, the manner of applying them to Practice, in Caſes of this Nature, is not Perfect; it being only by Tryals that I came to diſcover that $2 + \sqrt{-3}$ is the Cube Root of $-10 + \sqrt{-243}$: I will therefore proceed firſt to the Diſcovery of that General *Theorem*, from which *Cardan's* are deducible, and then ſhew how Æquations, in which the Value of $\frac{rr}{4} + \frac{b^3}{27}$ is Negative, may be thereby ſolv'd.

Having proceeded as far as the 16th Step Incluſive of the foregoing Operation, you'll from the 7th and 15th find for the

	17	$\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}+y^3=c$
$17-\sqrt{\frac{cc}{4}+\frac{b^3}{27}}-\frac{c}{2}$	18	$y^3=\frac{c}{2}-\sqrt{\frac{cc}{4}+\frac{b^3}{27}}$
18 ω 3	19	$y=\sqrt[3]{\frac{c}{2}-\sqrt{\frac{cc}{4}+\frac{bbb}{27}}}$
From 16, 19 and 4	20	$\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}}+$ $\sqrt[3]{\frac{c}{2}-\sqrt{\frac{cc}{4}+\frac{b^3}{27}}}=z=x^{\frac{n}{3}}$
20 ω $\frac{n}{3}$	21	$\sqrt[\frac{n}{3}]{\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}+\frac{b^3}{27}}}+\sqrt[3]{\frac{c}{2}-\sqrt{\frac{cc}{4}+\frac{b^3}{27}}}}=x.$

SCHOLIUM.

From the 3d and 20th Steps *Cardan*'s *Theorems* may be eaſily deduced.

Example. *In which the Value of* $\frac{cc}{4}+\frac{b^3}{27}$ is $\sqsupset 0$.

If $a^3-12{,}03\,a=16$;

That is, If $a^3-pa=q$; and that $\frac{qq}{4}-\frac{p^3}{27}$ is $\sqsupset 0$;

'Tis required to find one of the Values of a *proximè* by the help of *Cardan*'s Method.

The particular *Theorem* for evolving this Æquation, deducible from the 20th Step next foregoing, is

$$a=\sqrt[3]{\frac{q}{2}+\sqrt{\frac{qq}{4}-\frac{p^3}{27}}}+\sqrt[3]{\frac{q}{2}-\sqrt{\frac{qq}{4}-\frac{p^3}{27}}}$$

By the help of which, one of the Values of a may be found very near thus:

Suppoſe $\frac{q}{2} = r$, and $\sqrt{\frac{qq}{4} - \frac{p^3}{27}} = \sqrt{-s}$; then the Value of *a*, in the ſaid *Theorem*, will become $= \sqrt[3]{r + \sqrt{-s}} + \sqrt[3]{r - \sqrt{-s}}$: and (by *Chap. I. Part XV.*) $\sqrt[3]{r \pm \sqrt{-s}} = \overline{r}|^{\frac{1}{3}} + \frac{1}{3} \times \overline{r}|^{\frac{1}{3}-1} \times \pm\sqrt{-s} + \frac{\frac{1}{3}}{1} \times \frac{\frac{1}{3}-1}{2} \times \overline{r}|^{\frac{1}{3}-2} \times -s + \frac{\frac{1}{3}}{1} \times \frac{\frac{1}{3}-1}{2} \times \frac{\frac{1}{3}-2}{3} \times \overline{r}|^{\frac{1}{3}-3} \times \pm\sqrt{-s^3} + \frac{\frac{1}{3}}{1} \times \frac{\frac{1}{3}-1}{2} \times \frac{\frac{1}{3}-2}{3} \times \frac{\frac{1}{3}-3}{4} \times \overline{r}|^{\frac{1}{3}-4} \times + s^2$, &c. therefore $a \left(= \sqrt[3]{r + \sqrt{-s}} + \sqrt[3]{r - \sqrt{-s}}\right) = 2 \times \overline{r}|^{\frac{1}{3}} * + 2 \times \frac{\frac{1}{3}}{1} \times \frac{\frac{1}{3}-1}{2} \times \overline{r}|^{\frac{1}{3}-2} \times -s * + 2 \times \frac{\frac{1}{3}}{1} \times \frac{\frac{1}{3}-1}{2} \times \frac{\frac{1}{3}-2}{3} \times \frac{\frac{1}{3}-3}{4} \times \overline{r}|^{\frac{1}{3}-4} \times s^2$, &c. $= 2 \times \overline{r}|^{\frac{1}{3}} + \frac{2s}{9 \times \overline{r}|^{\frac{5}{3}}} - \frac{20ss}{243 \times \overline{r}|^{\frac{11}{3}}}$ &c. *Sine fine.*

Conſequently one of the Values of *a*, in the propoſ'd Æquation, is (becauſe $r = 8$, and $\sqrt{-s} = \sqrt{-.481201}$.) $= 2 \times \overline{8}|^{\frac{1}{3}} (= 2 \times 2) + \frac{.962402}{9 \times \overline{2}|^{5}} - \frac{4.631089}{243 \times \overline{2}|^{11}}$, &c. $= 4 + \frac{.962402}{288} - \frac{4.631089}{497664}$, &c. $= 4 + .003341$ &c. $- .000009$ &c. &c. $= 4.003332$ &c.

CHAP. III.

A Rule of *Des Cartes* for diſſolving a BIQUADRATICK, into two QUADRATICKS.

ANY Biquadratick Æquation, which hath the 2d Term, being propos'd to be diſſolv'd into two Quadraticks, muſt have its 2d Term firſt deſtroyed, (by *Chap.* 1. *Prop.* 1. *Schol.* 1.) and Let the Æquation thence produc'd, be ſuppopos'd equal to $x^4 * + qx^2 + rx + s = 0$; in which Æquation x^4 is ſuppos'd to have the Sign + prefix'd to it; but the Values of q, r or s, may either be Affirmative or Negative. Let us ſuppoſe this Æquation to be produc'd by the two following Quadraticks $x^2 + ex + f = 0$, and $x^2 - ex + g = 0$; in which Æquations x^2 is underſtood to have the Sign + prefix'd to it; and to the end that the ſecond Term ſhou'd be wanting in their Product, ex muſt have the Sign + in one Æquation, and − in the other, and the Values of f and g are to be determin'd in the following manner; then

$$x^4 + qx^2 + rx + s = x^4 \begin{matrix} +f \\ -e^2 \\ +g \end{matrix} x^2 \begin{matrix} -ef \\ +eg \end{matrix} x + fg;$$

And by Æquating their reſpective Terms, we have $q = f + g - e^2$, $r = \overline{g - f} \times e$, and $s = fg$; wherefore $q + e^2 = f + g$, $\frac{r}{e} = g - f$; conſequently $\frac{q + e^2 + \frac{r}{e}}{2} = g$, and $\frac{q + e^2 - \frac{r}{e}}{2} = f$, and the Product of the two laſt Steps is $\frac{q^2 + 2qe^2 + e^4 - \frac{rr}{ee}}{4} (= gf) = s$; which Æquation when reduc'd gives $e^6 + 2qe^4 \begin{matrix} +q^2 \\ -4s \end{matrix} e^2 = rr$. Now ſuppoſe $e^2 = y$, and then you'll reduce the foregoing Æquation to this Cubick one, $y^3 + 2qy^2 \begin{matrix}+qq\\-4s\end{matrix}y = r^2$, whoſe ſecond Term may be deſtroyed, and the Root of the Æquation thus had, extracted by the foregoing *Chap.* and conſequently the Value of y found; as alſo of $e = \sqrt{y}$; as alſo of $f = \dfrac{q + e^2 - \frac{r}{e}}{2}$, and laſtly of $g = \dfrac{q + e^2 + \frac{r}{e}}{2}$. And by extracting the Roots of the two Quadraticks $x^2 + ex + f = 0$, and $x^2 - ex + g = 0$, you'll have the four Roots of the Biquadratick $x^4 + qx^2 + rx + s = 0$; to wit, $x = -\frac{1}{2}e \pm \sqrt{\frac{ee}{4} - f}$, and $x = \frac{1}{2}e \pm \sqrt{\frac{ee}{4} - g}$, &c.

Example.

If $a^4 - 2a^3 - a + 2 = 0$, and the Values of a be required.

Firſt, In order to deſtroy the ſecond Term, ſuppoſe $a = x + \frac{1}{2}$, then the foregoing Æquation will become $x^4 * - \frac{3}{2}x^2 - 2x + \frac{21}{16} = 0$, in which $-\frac{3}{2} = q$, $-2 = r$, and $\frac{21}{16} = s$; wherefore $y^3 + 2qy^2 \begin{matrix}+qq\\-4s\end{matrix}y - rr \;(= 0) = y^3 - 3y^2 - 3y - 4 = 0$: And likewiſe to deſtroy the ſecond Term of this Æquation, ſuppoſe $y = v + 1$, then it will become $v^3 - 6v - 9 = 0$: Whence v will be found, by the laſt *Chap.* $= \sqrt[3]{4\frac{1}{2} + \sqrt{20\frac{1}{4} - 8}} + \sqrt[3]{4\frac{1}{2} - \sqrt{12\frac{1}{4}}} = \sqrt[3]{8} + \sqrt[3]{1} = 2 + 1 = 3$: And $v + 1 = y = 3 + 1 = 4$. Now y being thus found, its Square Root $= 2$ is $= e$, and $\dfrac{q + e^2 - \frac{r}{e}}{2}$; that is, $\dfrac{-\frac{3}{2} + 4 + 1}{2}$, or $1\frac{3}{4} = f$, and $\dfrac{q + e^2 + \frac{r}{e}}{2}$, or $\frac{3}{4} = g$: Wherefore the Æquations $x^2 + ex + f = 0$, and $x^2 - ex + g = 0$, are equal to $x^2 + 2x + 1\frac{3}{4} = 0$, and $x^2 - 2x + \frac{3}{4} = 0$ reſpectively; whence

$x =$

$x = -1 \pm \sqrt{-\frac{3}{4}}$ and $x = 1 \pm \sqrt{\frac{1}{4}} = \left\{\begin{matrix} 1\frac{1}{2} \\ \frac{1}{2} \end{matrix}\right\}$: But a is (by Suppoſition) $= x + \frac{1}{2}$; conſequently the 4 Values of a in the propos'd Æquation, are $-\frac{1}{2} + \sqrt{-\frac{3}{4}}$, $-\frac{1}{2} - \sqrt{-\frac{3}{4}}$, [illegible] $1\frac{1}{2}$ *and* $\frac{1}{2}$.

Note. *One* D. T. *In Act. Erudit. Lipſ. publiſhed a Method of Solving Æquations by deſtroying all the middle Terms; which, becauſe very tedious, and as to what relates to Cubicks, leſs Practicable in every Caſe than* Cardan's, *I ſhall here omit.*

CHAP. IV.

The Reſolution of ÆQUATIONS by *Stephinus*'s Method.

THE Roots of a given Æquation are by this Method found out by (ſometimes frequent) Tryals: Thus;

Example I.

Suppoſe the given Æquation to be $a^3 - 9a^2 + 26a = 24$; 'tis required to find the Values of a.

Firſt, I ſuppoſe $a = 1$, and working according to the Æquation, find that $a^3\ (1) - 9a^2\ (-9) + 26a\ (+26) = 18$; but it ought to be $= 24$; wherefore I conclude a is $\sqsubset 1$.

I try again, and ſuppoſe $a = 2$, then will $a^3\ (+8) - 9a^2$ (— [illegible] $+ 26a\ (+52) = 24$, which anſwers my Deſire, and gives me one real Value of a: after which I may divide the Æquation $a^3 - 9aa + 26a - 18 = 0$, by $a - 2 = 0$, which will bring it down to the Quadratick: Or I may proceed further in the ſame Method, and find alſo, that 3 and 4 are the two other real Roots.

Example II.

If this irregular Æquation was propos'd (where alſo the abſolute Number is a Fraction) $x^4 + 5x = 184638.6801$.

I can diſcover at firſt Sight almoſt, that a muſt be at leaſt $= 10$, and trying with 10, I find it too little; but trying with 100 I find that by much too great; proceeding again I find 30 too much; I try with 20, and find it by ſomethiug too ſmall, but 21 I find too big; wherefore I know that x muſt be = 20, with ſome Decimal Fraction annex'd; and at laſt I diſcover 20.7 to be the very Root ſought, or at leaſt one of the Roots of the given Æquation.

LEMMA to Chap. V.

How to find all the Diviſors of a given Number or Quantity.

RULE.

Divide the given Quantity or Number by the leaſt of its Diviſors that is $\sqsubset 1$, and the Quotient by the leaſt of its Diviſors that is $\sqsubset 1$, and ſo on till you have 1 for a Quotient, and you'll have all the prime Diviſors of the given Number or Quantity; then multiply each two, three, four, &c. of the prime Diviſors into themſelves continually, and the ſeveral Products are the compound Diviſors.

Examples.

I. So if all the Diviſors of 60 were required. Firſt divide it by 2, and the Quotient 30 by 2, and the Quotient 15 by 3, and the Quotient 5 by 5: then the prime Diviſors are 1, 2, 2, 3, 5; and the Compound ones produc'd by each two are 4, 6, 10, 15; by each three are 12, 20, 30; by each four (or by all) 60.

II. Again, If all the Diviſors of $21abb$ were required. Divide it by 3, and the Quotient $7abb$ by 7, and the Quotient abb by a, and the Quotient bb by b, and the Quotient b by b: then the prime Diviſors are 1, 3, 7, a, b, b; and the Compound ones produc'd by each two are 21, $3a$, $3b$, $7a$, $7b$, ab, bb; by each three, $21a$, $21b$, $3ab$, $3bb$, $7ab$, $7b^2$, abb; by each four, $21ab$, $21bb$, $3abb$, $7abb$; and by all five, $21abb$.

III. In like manner all the Diviſors of $2abb - 6aac$ are 1, 2, a, $bb - 3ac$; $2a$, $2bb - 6ac$, $abb - 3a^2c$, $2abb - 6aac$.

CHAP.

CHAP. V.

John Kerſey's Method of finding the Roots of ſome ADFECTED, CUBICK, BIQUADRATICK, *&c.* ÆQUATIONS.

FIRST prepare the given Æquation thus; *viz.*

1. If the Coefficient of the higheſt Power of the unknown Root be greater than 1, divide the whole Æquation by the ſaid Coefficient: And then,

2. If any of the Terms be Fractions, multiply the unknown Root by ſuch a Number as will give an Integer Product, and the Coefficient of its higheſt Power $= 1$. (By *Propoſition II. Chap. I.*) And the Root of the Æquation, (whether it be at firſt, or by theſe Directions thus prepar'd) call a. Then,

3. Reduce all the Terms of this Æquation to one ſide of it, and the other ſide will be 0.

Then find all the Diviſors of the abſolute Number in the Æquation ſo reduc'd; and try whether any of theſe Diviſors connected to the unknown Root a, by $-$ or $+$, will divide the total Sum of the reduc'd Æquation, without leaving a Remainder; for when ſuch Diviſion ſucceeds, either the known Part of the Binomial or Reſidual Diviſor, with a contrary Sign, is the deſired Value of the Root a, or at leaſt the Quotient gives an Æquation, whoſe firſt Term hath fewer Dimenſions by 1, than the Æquation divided; and if this Æquation contains three or more Dimenſions, let it be examined by Diviſion as before; and ſo on. By which Diviſions, the Roots of the given Æquation may be ſometimes made known, or the given Æquation ſometimes reduc'd to a Quadratick one; and then the ſought Root will be found by the Canons given for ſolving Adfected Quadraticks.

Example.

If $a^3 - 15a^2 + 74a - 120 = 0$. What are the Values of a?

The Diviſors of 120, the laſt Term, are 1, 2, 4, 8, 3, 6, *&c.* Then I try whether $a - 1$, $a + 1$, $a - 2$, $a + 2$, $a - 3$, or $a + 3$, will divide the given Æquation without leaving a Remainder: But finding that neither of them will do, I try next with $a - 4$, which will exactly do; and therefore 4 is one affirmative Value of the Root a; and the Quotient being $a^2 - 11a + 30 = 0$, the two other Roots will be found to be 5 and 6, by dividing $a^2 - 11a + 30 = 0$, by $a - 6$, or by $a - 5 = 0$, or by the Canon given for ſolving the 3d Caſe of Adfected Quadratick Æquations.

Example II.

Let it be required to find the Roots of this Æquation $a^3 - 36a^2 + 2a - 72 = 0$.

Its laſt Term can be exactly divided by 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, and 72; wherefore the propos'd Æquation is to be divided by $a +$ and $- 1$, $a +$ and $- 2$, &c. to find ſuch of them as will exactly do it; But ſince here are a great many Diviſors, and that (by the Compoſition of Æquations) there can be, at moſt, but three ſuch Diviſors, which will exactly divide the propos'd Æquation; you may try with a great many of thoſe Diviſors before you find any of the three ſought: * Wherefore, to ſave your ſelf a great deal of this Trouble, transform the given Æquation into another, each of whoſe Roots ſhall be more or leſs than thoſe of the propos'd one by a given Number, 1 is generally the moſt convenient; ſuppoſe therefore $a = x - 1$, then the propos'd Æquation will become $x^3 - 39x^2 + 77x - 111 = 0$, an Æquation whoſe Roots are each by 1 more than thoſe of the propos'd one, and the Diviſors of its laſt Term are 1, 3, 37, and 111: But ſince $a = x - 1$, it is evident that if any of the Values of x be 1, 3, 37, 111, or − 1, − 3, − 37, − 111, (as it, or they, muſt be, if it has any Rational one) that of a will be 0, 2, 36, 110, or − 2, − 4, − 38, − 112; but (by what was before ſaid) all the rational Values of a are inſerted among the following ones; *viz.* 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72, or − 1, − 2, − 3, − 4, &c. Conſequently, if a has any Rational Root in the propos'd Æquation, it, or they, muſt be 2, 36, or − 2, − 4: Wherefore you need now try to divide the propos'd Æquation only by $a - 2 = 0$, $a - 36 = 0$, $a + 2 = 0$, and $a + 4 = 0$: Wherefore I try firſt to divide it by $a - 2 = 0$; but that not exactly ſucceeding, I try next to divide it by $a - 36 = 0$, which exactly does, the Quotient being $a^2 + 2 = 0$, an Æquation wherein

* Mr. *Waeſſenaer's Method.*

wherein a is $= + \surd - 2$, and $- \surd - 2$; and conſequently the 3 Roots or Values of a, in the propos'd Æquation, are 36, $\surd - 2$, and $- \surd - 2$.

CHAP. VI.

The Solution of ÆQUATIONS, by Sir *Iſ. Newton*'s Method.

THERE is an univerſal Method of Extracting Roots, either in Numbers or Symbols, Invented by Sir *Iſ. Newton*, which you may find in *Pages* 381, 382, and 383 of Dr. *Wallis*'s *Algebra*, which is to this Effect.

Firſt, Find the firſt or greateſt Member of the Root ſought (y), and if it be not $= y$, let that Member $+ p$ be ſuppos'd $= y$; then having ſubſtituted this Binomial and its reſpective Powers for y, and its Powers in the Æquation, collect its ſeveral Terms into one Sum; then find the ſecond Member of the ſaid Root, which is done (*in ſome Caſes, but not in all in the beginning of the Operation*) by dividing the firſt Term (or abſolute Number or known Quantity) of the ſaid Sum, by the Coefficient of p in the ſecond Term thereof, and let the Quotient, affected with the contrary Sign to what it has $+ q$, be ſuppoſed $= p$; then proceed with this Binomial or Reſidual in reſpect of p, as you did with the other in reſpect of y: and ſo on.

Note, When you have found 3, 4, or more of the firſt Figures, or Members of the Root, you may find as many, or almoſt as many more, by dividing the firſt Term of the laſt Sum, by the Coefficient in the ſecond.

Example I.

If $y^3 - 2y - 5 = 0$; What is one of the Values y?

$$y^3 - 2y - 5 = 0 \left(\begin{array}{c} 2.1 - .00544852 = 2.09455148 \\ = y \text{ very near.} \end{array}\right\}$$

$$2 + p = y \;\therefore\; \left|\begin{array}{r} y^3 \\ -2y \\ -5 \end{array}\right. \begin{array}{c} = \\ = \\ = \end{array} \begin{array}{r} 8 \\ -4 \\ -5 \end{array} \begin{array}{l} \left.\begin{array}{r} +12 \\ -2 \end{array}\right\} p + 6p^2 + p^3 \\ \\ \end{array}$$

$$\text{Sum} \quad -1 + 10p + 6p^2 + p^3$$

$$.1 + q = p \;\therefore\; \left|\begin{array}{r} p^3 \\ 6p^2 \\ 10p \\ -1 \end{array}\right. \begin{array}{c} = \\ = \\ = \\ = \end{array} \begin{array}{r} .001 \\ .06 \\ 1. \\ -1 \end{array} \begin{array}{l} \left.\begin{array}{r} +.03 \\ +1.2 \\ +10 \end{array}\right\} \times q \begin{array}{l} \left.\begin{array}{r} +.3 \\ +6. \end{array}\right\} \times qq + q^3 \\ \\ \end{array} \\ \\ \end{array}$$

$$\text{Sum} \quad .061 + 11.23\,q + 6.3\,q^2 + q^3.$$

$$-.0054 + r = q \;\therefore\; \left|\begin{array}{r} q^3 \\ 6.3q^2 \\ 11.23q \\ +.061 \end{array}\right. \begin{array}{c} = \\ = \\ = \\ = \end{array} \begin{array}{r} -.0000001 \\ .0001837 \\ -.060642 \\ .061 \end{array} \begin{array}{l} \left.\begin{array}{r} +.000 \\ -.068 \\ +11.23 \end{array}\right\} \begin{array}{l} \text{\&c.} \\ \times r \\ \\ \end{array} \\ \\ \end{array}$$

$$\text{Sum} \quad + .0005416 + 11.162r, \text{ \&c.}$$

$$-.00004852 + s = r \,|$$

Example II.

If $y^3 + axy + a^2y - x^3 - 2a^3 = 0$: *Quære* y *proximè.*

$$y^3 + axy + a^2y - x^3 - 2a^3 = 0 \;\left(a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2}\right., \text{ \&c. } \textit{Sine fine.}$$

$$a+p=y \because \quad \begin{array}{rcl} y^3 &=& a^3 + 3a^2p + 3ap^2 + p^3 \\ axy &=& a^2x + axp \\ a^2y &=& a^3 + a^2p \\ -x^3 &=& -x^3 \\ -2a^3 &=& -2a^3 \end{array}$$

$$\text{Sum} \left.\begin{array}{l} aax + 4a^2 \\ -x^3 + ax \end{array}\right\} \times p + 3ap^2 + p^3$$

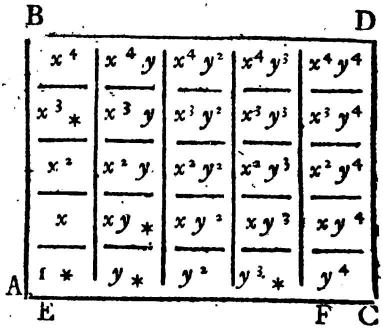
$$-\frac{x}{4}+q=p \because \quad \begin{array}{rcl} p^3 &=& -\tfrac{1}{64}x^3 + \tfrac{3}{16}xx \\ 3ap^2 &=& \tfrac{3}{16}ax^2 - \tfrac{3}{2}ax \\ axp &=& -\tfrac{1}{4}ax^2 + ax \\ 4a^2p &=& -a^2x + 4a^2 \\ a^2x &=& a^2x \\ -x^3 &=& -x^3 \end{array} \left.\begin{array}{l} \&c. \\ \times q, \\ \\ \end{array}\right.$$

$$\text{Sum} \quad \begin{array}{l} -\tfrac{1}{16}ax^2 \\ -\tfrac{65}{64}x^3 \end{array} \left.\begin{array}{l} + 4a^2 \\ - \tfrac{1}{2}ax \\ + \tfrac{3}{16}x^2 \end{array}\right\} q, \ \&c.$$

$$\frac{x^2}{64a} + \frac{131x^3}{512a^2} + r = q.$$

The greateſt Difficulty in this *Example* II. is to find the firſt Member of the Root, which is done thus.

Let a Parallelogram ABCD be (or ſuppos'd to be) deſcrib'd; and let it be divided into as many ſimilar ſmall ▭s as are requiſit; then Denominate each of theſe ſmall ▭s from the Dimenſions of the two indefinite Quantities of the Æquation, as x and y increaſing regularly (as in the annexed Figure) from the ſmall ▭ 1, of which y denotes the Root to be extracted, and x the other indefinite Quantity. Then mark each ſmall ▭, which anſwers the Terms of the given Æquation



x^4	x^4y	x^4y^2	x^4y^3	x^4y^4
x^3 *	x^3y	x^3y^2	x^3y^3	x^3y^4
x^2	x^2y	x^2y^2	x^2y^3	x^2y^4
x	xy *	xy^2	xy^3	xy^4
1 *	y *	y^2	y^3 *	y^4

quation with an Aſteriſk; and at the loweſt Corner (or Angle) to the left Hand of the loweſt ſmall ▭ to the left Hand thus mark'd as a Center, fix the end E of the ſtraight Line (or Ruler) E F, and turn the other end F, from A, towards C, till the Line E F touches one or more of the ſmall ▭s thus mark'd, and with the Terms of the Æquation anſwering them of the ſmall ▭s thus touch'd, make a ſuppos'd Æquation.

Thus the propos'd Æquation $y^3 + axy + a^2 y - x^3 - 2a^3 = 0$, exhibits $-2a^3 + a^2 y + y^3 = 0$; and therefore you have $y = a$ for the firſt Member of the Root to be extracted. *&c.*

PART XIII.

How to raiſe Canons for finding the Sums of the Powers of an Arithmetical Progreſſion Continued.

THEOREM I.

IN a Series of Units (as 1, 1, 1, 1, &c.) if the Number of Terms (n) be multiplied by either of them, the Product (n) will be equal the Sum of all the Terms in the ſaid Series.

This is evident from the nature of Multiplication.

THEOREM II.

In a Series of Numbers in an Arithmetical Progreſſion increaſing, whoſe firſt Term is = to its common Exceſs = 1, and number of Terms = n (as 1, 2, 3, 4, &c. and n); if to the laſt Term (n) you add the firſt (1), and multiply the Sum by half the Number of Terms $\left(\frac{n}{2}\right)$, the Product $\left(\frac{n^2+n}{2}\right)$ is equal to the Sum of the ſaid Series.

This has been Demonſtrated in Arithmetical Progreſſion. See 97 pages where you have $s = na + \frac{nne-ne}{2} =$ (in this Case) $\frac{n^2+n}{2}$.

THEOREM III.

In a Series of Squares, whoſe Sides or Roots are in an Arithmetick Progreſſion increaſing, whoſe firſt Term and common Exceſs are each $= 1$, and number of Terms $= n$, (as 1^2, 2^2, 3^2, *&c.* and n^2) I ſay that $\frac{n^3 + \frac{3}{2}n^2 + \frac{1}{2}n}{3}$ is $=$ to their Sum $(z) = 1 + 4 + 9 + 16 + \&c. + nn$.

DEMONSTRATION.

It is manifeſt that z is $=$ to the Difference of the Sums of the two next following Series, each of whoſe Ranks is in an Arithmetical Progreſſion continued.

Sum of the Greater Series. *Sum of the Leſſer Series.*

$$= \left\{\begin{array}{c} \underbrace{\begin{array}{l} 1 + 2 + 3 + 4 + \&c. + n \\ 1 + 2 + 3 + 4 + \&c. + n \\ 1 + 2 + 3 + 4 + \&c. + n \\ 1 + 2 + 3 + 4 + \&c. + n \end{array}} \\ \text{\&c. to } n \text{ Ranks continued.} \end{array}\right\} - \left\{\begin{array}{c} \underbrace{\begin{array}{l} 0 \\ 1 \\ 1 + 2 \\ 1 + 2 + 3 \end{array}} \\ \text{\&c. to } n \text{ Ranks continued.} \end{array}\right\}$$

Difference of the Sums of the Greater and Leſſer Series.

$$= \left\{\begin{array}{c} \underbrace{\begin{array}{r} 1 + 2 + 3 + 4 + \&c. + n \\ 2 + 3 + 4 + \&c. + n \\ 3 + 4 + \&c. + n \\ 4 + \&c. + n \end{array}} \\ \text{\&c. to } n \text{ Ranks continued.} \end{array}\right\}$$

For $1 = 1^2$; $2 + 2 = 2^2$; $3 + 3 + 3 = 3^2$; $4 + 4 + 4 + 4 = 4^2$, *&c.*

And $1+2+3+4+\&c.+n$ (or the Sum of 1, 2, 3, 4, *&c.* continued to n Terms) is (by *Theorem* the 2d.) $=\frac{nn+n}{2}$; Wherefore the Sum of the Greater Series (being $=n$ times $1+2+3+4+\&c.+n$) is $=\frac{n^3+n^2}{2}$.

In the next Place we are to find the Sum of the Leſſer Series; in order to which conſider its 1ſt Rank being 0, its 2d Rank 1, its 3d Rank $1+2$, its 4th Rank $1+2+3$, &c. that therefore

Its nth Rank muſt be $=1+2+3+4+\&c.+\overline{n-1}$

Alſo its $\overline{n-1}$th Rank $=1+2+3+4+\&c.+\overline{n-2}$

Again its $\overline{n-2}$th Rank $=1+2+3+4+\&c.+\overline{n-3}$

&c. See Page 206.

Therefore the Sum of the nth Rank of the ſaid Leſſer Series is (by ſubſtituting $\overline{n-1}$, for n in the Canon inſerted in *Theorem II.*) $=\frac{\overline{n-1}|^2+\overline{n-1}}{2}$: Alſo the Sum of the $\overline{n-1}$th Rank is $=\frac{\overline{n-2}|^2+\overline{n-2}}{2}$: Again the Sum of the $\overline{n-2}$th Rank is $=\frac{\overline{n-3}|^2+\overline{n-3}}{2}$, &c. Wherefore

$$\frac{\overline{n-1}|^2+\overline{n-1}}{2}+\frac{\overline{n-2}|^2+\overline{n-2}}{2}+\frac{\overline{n-3}|^2+\overline{n-3}}{2}$$

$+$ &c. is equal to the Sum of the aforeſaid Leſſer Series: But $\frac{\overline{n-1}|^2+\overline{n-2}|^2+\overline{n-3}|^2+\&c.+1}{2}={}^{*}\frac{z}{2}-\frac{n^2}{2}$; and $\frac{\overline{n-1}+\overline{n-2}+\overline{n-3}+\&c.+1}{2}$ is (by *Theorem II.*) $=\frac{nn-n}{4}$ #; wherefore the ſaid Sum of the Leſſer Series is $=\frac{z^2}{2}-\frac{n^2}{2}+\frac{nn-n}{4}$, conſequently the Difference of the Sums

* by Hypoth. and Transp. n^2, and divid. by 2.

# because (by 2 Theor.) $n+\overline{n-1}+\overline{n-2}+\overline{n-3}+\&c.+1$ is $=$ whence by Transp. n, and divid. by 2, you have &c.

Sums of the Greater and Leſſer Series is $= \frac{n^3+n^2}{2} - \frac{z}{2} + \frac{nn}{2} - \frac{nn-n}{4} = z$ (by Suppoſition) $= \frac{2n^3+3nn+n}{4} - \frac{z}{2}$; and by Tranſpoſing $-\frac{z}{2}$, we have $\frac{3}{2}z = \frac{2n^3+3n^2+n}{4}$: and finally by multiplying each Part by $\frac{2}{3}$, we will have $z = \frac{2n^3+3n^2+n}{6} = \frac{n^3+1\frac{1}{2}n^2+\frac{1}{2}n}{3}$.

Q. E. D.

THEOREM IV.

In a Series of Cubes, whoſe Roots are in an Arithmetick Progreſſion Increaſing, whoſe firſt Term = common Exceſs is 1, and number of Terms is $= n$ (as $1^3, 2^3, 3^3, 4^3$, &c. n^3) I ſay that $\frac{n^4+2n^3+n^2}{4}$ is equal to their Sum (z); that is equal to $1+8+27+64+$ &c. $+n^3$.

DEMONSTRATION.

The ſaid Sum z is manifeſtly equal to the Difference of the Sums of the Greater and Leſſer following Series, each of whoſe Ranks is the Sum of the Squares of an ÷

Sum of the Greater Series. *Sum of the Leſſer Series.*

$$= \left[\underbrace{\begin{array}{l} 1+4+9+16+\&c.+n^2 \\ 1+4+9+16+\&c.+n^2 \\ 1+4+9+16+\&c.+n^2 \\ 1+4+9+16+\&c.+n^2 \end{array}}_{\&c.\ \text{to}\ n\ \text{Ranks continued.}} \right] - \left[\underbrace{\begin{array}{l} 0 \\ 1 \\ 1+4 \\ 1+4+9 \end{array}}_{\&c.\ \text{to}\ n\ \text{Ranks continued.}} \right]$$

Difference of the Sums of the two next foregoing Series.

$$= \left\{ \underbrace{\begin{array}{r} 1+4+9+16+\&c.+n^2 \\ 4+9+16+\&c.+n^2 \\ 9+16+\&c.+n^2 \\ 16+\&c.+n^2 \end{array}}_{\text{\&c. to } n \text{ Ranks continued.}} \right\}$$

For $1 = 1^3$, $4+4 = 2^3$, $9+9+9 = 3^3$, &c. And $1+4+9+16+\&c.+nn$ is, (by *Theorem III.*) $= \frac{2n^3+3n^2+n}{6}$; wherefore the Sum of the foregoing Greater Series (being equal to n times $1+4+9+16+\&c.+n^2$) is $= \frac{2n^4+3n^3+n^2}{6}$.

Again in order to find the Sum of the Leſſer Series, conſider, 0 being the firſt Rank, 1 the 2d Rank, $1+2^2$ the 3d Rank, $1+2^2+3^2$ the 4th Rank, &c. of the ſaid Leſſer Series; that therefore the nth Rank thereof will be $= 1+2^2+3^2+4^2+\&c.+\overline{n-1}|^2$; alſo the $\overline{n-1}$ th Rank will be $= 1+2^2+3^2+4^2+\&c.+\overline{n-2}|^2$: Again the $\overline{n-2}$th Rank will be $= 1+2^2+3^2+4^2+\&c.+\overline{n-3}|^2$: &c. Therefore the Sum of the nth Rank of the ſaid Leſſer Series is (by *Theorem III.*) $= \frac{2\overline{n-1}|^3+3\overline{n-1}|^2+\overline{n-1}}{6}$. *Alſo the Sum of the $\overline{n-1}$ th Rank is $= \frac{2\overline{n-2}|^3+3\overline{n-2}|^2+\overline{n-2}}{6}$: Again the Sum

* by substituting n-1 for n in 3^d Theor.

of the $\overline{n-2}$th Rank is $=\frac{2\overline{n-3}|^3+3\overline{n-3}|^2+\overline{n-3}}{6}$, &c. Wherefore $\frac{2\overline{n-1}|^3+3\overline{n-1}|^2+\overline{n-1}}{6}+\frac{2\overline{n-2}|^3+3\overline{n-2}|^2+\overline{n-2}}{6}+\frac{2\overline{n-3}|^3+3\overline{n-3}|^2+\overline{n-3}}{6}$ + &c. is equal to the Sum of the ſaid Leſſer Series: But $\frac{2\overline{n-1}|^3+2\overline{n-2}|^3+2\overline{n-3}|^3+\&c.+2\times1}{6}$ is $=\frac{z}{3}-\frac{n^3}{3}$*: Alſo∧ $\frac{3\overline{n-1}|^2+3\overline{n-2}|^2+3\overline{n-3}|^2+\&c.+3\times1}{6}$ is (by *Theorem III.*) $=\frac{2\overline{n-1}|^3+3\overline{n-1}|^2+\overline{n-1}}{12}=\frac{2n^3-3n^2+n}{12}$: and $\frac{\overline{n-1}+\overline{n-2}+\overline{n-3}+\&c.+1}{6}$ is (by *Theorem II.*) $=\frac{nn-n}{12}$#; wherefore the Sum of the aforeſaid Leſſer Series is $=\frac{z}{3}-\frac{n^3}{3}+\frac{2n^3-3n^2+n+nn-n}{12}=\frac{z}{3}-\frac{2n^3+2nn}{12}$; wherefore the Difference of the Sums of the Greater and Leſſer Series is $=\frac{2n^4+3n^3+n^2}{6}-\frac{z}{3}+\frac{2n^3+2n^2}{12}=\frac{n^4+2n^3+n^2}{3}-\frac{z}{3}=z$ (by Suppoſition); and by tranſpoſing $-\frac{z}{3}$ we have $\frac{4}{3}z=\frac{n^4+2n^3+n^2}{3}$: and by multiplying each Part by $\frac{1}{4}$, we have $z=\frac{n^4+2n^3+n^2}{4}$. *Q. E. D.*

because by 2 Theor. $n+\overline{n-1}+\overline{n-2}+\overline{n-3}+\&c.+1$ is $=\frac{\quad}{2}$.

d 2

* because (by hyp.) $n^3+\overline{n-1}|^3+\overline{n-2}|^3+\overline{n-3}|^3+\&c.+1$ is $=z$.

∧ (because by 3 Theor. $\overline{n-1}|^2+\overline{n-2}|^2+\overline{n-3}|^2+\&c.+1$ is $=\underline{2\overline{n-1}|^3+3\overline{n-2}|^2+n}$) there is

THEO-

THEOREM V.

In a Series of Biquadrats, whoſe Roots are in an Arithmetick Progreſſion increaſing, whoſe firſt Term is $= 1 =$ common Exceſs, and number of Terms $= n$ (as 1^4, 2^4, 3^4, 4^4, &c. n^4); I ſay that $\frac{n^5 + 2\frac{1}{2}n^4 + 1\frac{2}{3}n^3 - \frac{1}{6}n}{5}$ is equal to their Sum (z).

DEMONSTRATION.

The ſaid Sum z is manifeſtly equal to the Difference of the Sums of the two next following Series, each of whoſe Ranks is the Sum of the Cubes of an $\div$

Sum of the Greater Series. *Sum of the Leſſer Series.*

$$= \left\{ \begin{array}{l} 1+8+27+64+\&c.+n^3 \\ 1+8+27+64+\&c.+n^3 \\ 1+8+27+64+\&c.+n^3 \\ \underbrace{1+8+27+64+\&c.+n^3}_{\text{\&c. to } n \text{ Ranks continued.}} \end{array} \right\} - \left\{ \begin{array}{l} 0 \\ 1 \\ 1+8 \\ \underbrace{1+8+27}_{\text{\&c. to } n \text{ Ranks continued.}} \end{array} \right\}$$

Difference of the Sums of the two foregoing Series.

$$= \left\{ \begin{array}{r} 1+8+27+64+\&c.+n^3 \\ 8+27+64+\&c.+n^3 \\ 27+64+\&c.+n^3 \\ \underbrace{64+\&c.+n^3}_{\text{\&c. to } n \text{ Ranks continued.}} \end{array} \right\}$$

For $1 = 1^4$; $8 + 8 = 2^4$; $27 + 27 + 27 = 3^4$; &c. And $1 + 8 + 27 + 64 + \&c. + n^3$, is (by *Theorem IV.*) $= \frac{n^4 + 2n^3 + n^2}{4}$; therefore the Sum of the foregoing Greater Series (being $=$ *n* times $1 + 8 + 27 + 64 + \&c. + n^3$) is $= \frac{n^5 + 2n^4 + n^3}{4}$.

Now for to find the Sum of the Leſſer Series, conſider, ſince 0 is the firſt Rank thereof, 1 the 2d Rank, $1^3 + 2^3$ the third Rank, $1^3 + 2^3 + 3^3$ the fourth Rank, &c. that $1^3 + 2^3 + 3^3 + 4^3 + \&c.\ \overline{n-1}|^3$ will be $=$ *n*th Rank; alſo $1^3 + 2^3 + 3^3 + 4^3 + \&c. + \overline{n-2}|^3 =$ the $\overline{n-1}$ th Rank, alſo $1 + 2^3 + 3^3 + 4^3 + \&c. + \overline{n-3}|^3$ $=$ the $\overline{n-2}$ th Rank &c. of the ſaid Leſſer Series; wherefore the Sum of the *n*th Rank of the ſaid Leſſer Series is (by *Theorem IV.*) $= \frac{\overline{n-1}|^4 + 2\,\overline{n-1}|^3 + \overline{n-1}|^2}{4}$: Alſo the Sum of the $\overline{n-1}$ th Rank $= \frac{\overline{n-2}|^4 + 2\,\overline{n-2}|^3 + \overline{n-2}|^2}{4}$ Again, the Sum of the $\overline{n-2}$ th Rank of the ſaid Leſſer Series, is (by *Theorem IV.*) $= \frac{\overline{n-3}|^4 + 2\,\overline{n-3}|^3 + \overline{n-3}|^2}{4}$: &c. therefore

$$\frac{\overline{n-1}|^4 + \overline{n-2}|^4 + \overline{n-3}|^4 + \&c. + 1}{4}$$

$$+ \frac{2\,\overline{n-1}|^3 + 2\,\overline{n-2}|^3 + 2\,\overline{n-3}|^3 + \&c. + 2 \times 1}{4}$$

$$+ \frac{\overline{n-1}|^2 + \overline{n-2}|^2 + \overline{n-3}|^2 + \&c. + 1,}{4}$$ is $=$ to the Sum of the ſaid Leſſer Series:

But $\frac{\overline{n-1}|^4 + \overline{n-2}|^4 + \overline{n-3}|^4 + \&c. + 1}{4} = \frac{z}{4} - \frac{n^4}{4}$

Alſo $\frac{2\times\overline{n-1}|^3+\overline{n-2}|^3+\overline{n-3}|^3+\&c.+1}{4}$ is (by *Theorem IV.*) $=\frac{2\times\overline{n-1}|^4+2\,\overline{n-1}|^3+\overline{n-1}|^2}{16}=\frac{n^4-2n^3+n^2}{8}$.

Alſo $\frac{\overline{n-1}|^2+\overline{n-2}|^2+\overline{n-3}|^2+\&c.+1}{4}$ is (by *Theorem III.*) $=\frac{2\,\overline{n-1}|^3+3.\overline{n-1}|^2+\overline{n-1}}{24}=\frac{2n^3-3n^2+n}{24}$; wherefore the Sum of the ſaid Leſſer Series is $=\frac{z}{4}-\frac{n^4}{4}+\frac{n^4-2n^3+n^2}{8}+\frac{2n^3-3n^2+n}{24}$; conſequently the Difference of the Sums of the ſaid Greater and Leſſer Series is $=\frac{n^5+2n^4+n^3}{4}-\frac{z}{4}+\frac{n}{4}-\frac{n^4-2n^3+n^2}{8}-\frac{2n^3-3n^2+n}{24}=\frac{6n^5+15n^4+10n^3 * -n}{24}-\frac{z}{4}=z$ (by Suppoſition); and by tranſpoſing $-\frac{z}{4}$ and then multiplying each Part by $\frac{4}{5}$ we will have $\frac{n^5+2\frac{1}{2}n^4+1\frac{2}{3}n^3 * -\frac{1}{6}n}{5}=z$. *Q. E. D.*

&c.

See page 200
Here n is = 7, n−1 = 6, n−2 = 5,
n−3 = 4, &c.

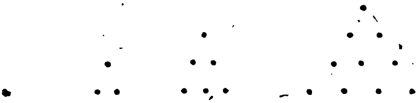
0
1
$\overline{n-4}^{th}$ Rank --- 1+2
$\overline{n-3}^{th}$ Rank --- 1+2+3 = $\overline{n-4}^{th}$ + $\overline{n-4}$
$\overline{n-2}^{th}$ Rank --- 1+2+3+4 = $\overline{n-3}^{th}$ Rank + $\overline{n-3}$
$\overline{n-1}^{th}$ Rank --- 1+2+3+4+5 = $\overline{n-2}^{th}$ Rank + $\overline{n-2}$
n^{th} Rank --- 1+2+3+4+5+6 = $\overline{n-1}^{th}$ Rank + $\overline{n-1}$

PART

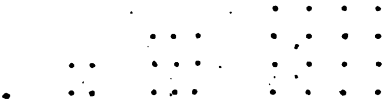
PART XIV.

Of Polygonal Numbers.

POlygonal or Multangular Numbers are the Sums or Aggregats of a rank of Numbers in Arithmetical Progreſſion continued from Unity, and are ſo called, becauſe they repreſent the number of Points that are required to fill ſuch regular Polygons at equal Diſtances or Lines drawn Parallel to the ſides of the Figure: as



&c. are Triangulars.



&c. are Quadrangulars.

&c.

And are thus formed.

Numbers in Arithmetical Progreſſion.	*Whoſe common Difference is*	*Sums added from* 1.	*Polygon.*
1, 2, 3, 4, &c.	1	1, 3, 6, 10, &c.	Triang.
1, 3, 5, 7, &c.	2	1, 4, 9, 16, &c.	Quadr.
1, 4, 7, 10, &c.	3	1, 5, 12, 22, &c.	Pentang.
1, 5, 9, 13, &c.	4	1, 6, 15, 28, &c.	Hexang.
&c.	&c.	&c.	&c.

Hence, by Inſpection, theſe two Obſervations are evident;

1. That the common Difference will be always leſs, by 2, than the Number of Angles.

2. That the Side of the Polygon is always equal to the Number of Terms, that compoſe it.*
of the arithmetical Progression

Therefore putting d = Common Difference.
n = Number of Terms.
l = Laſt, or Greateſt Term.
z = Sum of all the Terms.
p = Polygon.
a = Number of Angles.

$$\text{We have } p = \frac{\overline{nn - n} \times d}{2} + n.$$

DEMONSTRATION.

Since every Polygon is the Aggregate of Numbers in ÷, whoſe firſt Term is 1, (by *Part VIII. Chap.* 1. *Schol.* 1.) $p = \frac{\overline{1 + l} \times n}{2}$, and (by *Theorem III.* of the ſaid Chapter,) $l = 1 + nd - d$, which ſubſtitute in the former Value of p, and you'll have $\frac{\overline{1 + 1 + nd - d} \times n}{2} = p = \frac{n^2 d - nd}{2} + n$

Q. E. D.

* This is the Definition of the Side of a Polygon: So 2 is the Side of the Triangular 3, 3 that of the Triangular 6; 2 is the Side of the Quadrangular 4, 3 that Quadrangular 9; 4 is the Side [illegible] 22.

COROL-

COROLLARY.

And becauſe d is given, (by *Obſervation* 1.) if its Value be ſubſtituted in this General *Theorem*, we may deduce Particular ones for each Polygon: As in a

$$\left.\begin{array}{r}\text{Triang.}\\ \text{Quadrang.}\\ \text{Pentang.}\\ \text{Hexang.}\end{array}\right\} d\text{ is}=\left\{\begin{array}{l}1\\ 2\\ 3\\ 4\\ \text{\&c.}\end{array}\right\}\text{and } p=\frac{n^2d-nd}{2}+n\;\Big\}\;\text{Therefore } p=\left\{\begin{array}{l}\frac{nn+n}{2}\\ nn\\ \frac{3n^2-n}{2}\\ 2n^2-n\end{array}\right.$$

SCHOLIUM.

The common Difference in a Rank or Series of Polygons is always $=\frac{nn-n}{2}$, which have the ſame Side, or are compos'd of the ſame Number of Terms; therefore by Virtue of the ſaid *Theorem* 1. the Sum of any ſuch Series or Rank of Polygonal Numbers will always be $=\frac{an+ap-2n-2p}{2}$, putting p = Greateſt or Laſt Term, and a = Angles of it; then $a-2$ = Number of the Terms, and n = firſt or leaſt Term. And if we ſubſtitute a in Numbers in $\frac{an+ap-2n-2p}{2}$, we have particular *Theorems* for any Series: as in a

$$\left.\begin{array}{r}\text{Triang.}\\ \text{Quadrang.}\\ \text{Pentang.}\\ \text{Hexang.}\end{array}\right\} = a = \left\{\begin{array}{l}3\\ 4\\ 5\\ 6\end{array}\right\}\begin{array}{c}\text{and Sum} = \\ \frac{an+ap-2n-2p}{2}\\ \\ \text{\&c.}\end{array}\Big\}\;\text{Therefore the ſaid Sum}=\left\{\begin{array}{l}\frac{n+p}{2}\\ n+p\\ \frac{3n+3p}{2}\\ 2n+2p\end{array}\right.$$

But becauſe n may not be always given, ſince we find $p = \frac{n^2d - nd + 2n}{2}$; n will therefore be found (by *Part X.*) $= \frac{d - 2 + \sqrt{dd - 4d + 4 + 8dp}}{2d}$. And becauſe d is given (by *Obſ.* 1.) if we ſubſtitute its Value, we ſhall have particular Theorems in this Caſe likewiſe; *viz.* in a

$$\left.\begin{array}{l}\text{Triang.}\\ \text{Quadrang.}\\ \text{Pentang.}\\ \text{Hexang.}\end{array}\right\} d = \left\{\begin{array}{c}1\\ 2\\ 3\\ 4\\ \text{\&c.}\end{array}\right\} \text{and } n = \frac{d - 2 + \sqrt{d^2 - 4d + 4 + 8dp}}{2d}$$

$$\text{Therefore } n = \left\{\begin{array}{c}\frac{-1 + \sqrt{1 + 8p}}{2}\\ \frac{0 + \sqrt{0 + 16p}}{4}\\ \frac{1 + \sqrt{1 + 24p}}{6}\\ \frac{2 + \sqrt{4 + 32p}}{8}\\ \text{\&c.}\end{array}\right.$$

Thus having ſhew'd how from the Side given to find the Polygon, or from the Polygon to find the Side; as alſo the Sum of any Series, or Rank of regular Polygons from a Lateral: I ſhall now give an univerſal Theorem for finding any Triangular, firſt Pyramidal, or other Number under-mentioned: as See what is written at the end of the Book.

Units	1.	1.	1.	1.	1 .	1			1ſt	
Laterals	1.	2.	3.	4.	5 .	6			2d	
Triangulars	1.	3.	6.	10.	15 .	21	&c.	Or the Figurate Numbers of the	3d	Order.
1ſt Pyramid.	1.	4.	10.	20.	35 .	56			4th	
2d Pyramid.	1.	5.	15.	35.	70 .	126			5th	
3d Pyramid.	1.	6.	21.	56.	126 .	252			6th	

&c.

OBSER-

OBSERVATION I.

Here 'tis evident that each figurate Number is the aggregate of the preceding Series ſo far, or of the preceeding figurat Number, and that above it ſelf.

OBSERVATION II.

It is alſo evident if b and c be ſuppos'd = any two whole Numbers, that b figurate Number of c Order, is equal to c figurate Number of b Order.

LEMMA.

If n be = the ſide of a figurate Number of ſuch an Order as that 1, $1 \times n$, $1 \times \frac{n}{1} \times \frac{n+1}{2}$, $1 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, $1 \times \frac{n+0}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, &c. i. e. (Since $\frac{a}{e} \times \frac{z}{y} = \frac{z}{e} \times \frac{a}{y}$) 1, $1 \times n$, $1 \times \frac{n+1}{1} \times \frac{n}{2}$, $1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n}{3}$, $1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times \frac{n}{4}$, &c. be = the reſpective figurate Numbers of that Order; and if m be $= n+1$ the Side of a figurat Number of the next Greater Order: I ſay, that 1, $1 \times m$, $1 \times \frac{m+0}{1} \times \frac{m+1}{2}$, $1 \times \frac{m+0}{1} \times \frac{m+1}{2} \times \frac{m+2}{3}$, $1 \times \frac{m}{1} \times \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{m+3}{4}$, &c. ſhall be = the reſpective figurate Numbers of that Order, whoſe Side is m.

DEMONSTRATION.

It is evident (by the nature of Multiplication) that $1 = 1$, $1 \times n + 1 = 1 \times n + 1$, $1 \times \frac{n+1}{1} \times \frac{n}{2} + 1 \times n + 1 = 1 \times \frac{n+1}{1} \times \overline{\frac{n}{2} + 1} \left(\frac{n+2}{2}\right)$, $1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n}{3} + 1 \times$

$\frac{n+1}{1} \times \frac{n+2}{2} = 1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \overline{\frac{n}{3} + 1} \left(\frac{n+3}{3}\right)$, $1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times \frac{n}{4} + 1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} = 1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times \overline{\frac{n}{4} + 1} \left(\frac{n+4}{4}\right)$, &c. Therefore, if $1, n, 1 \times \frac{n+1}{1} \times \frac{n}{2}, 1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n}{3}, 1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times \frac{n}{4}$, &c. be = the reſpective figurate Numbers of the Order whoſe ſide is n,

$1, 1 \times n+1, 1 \times \frac{n+1}{1} \times \frac{n+2}{2}, 1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3}, 1 \times \frac{n+1}{1} \times \frac{n+2}{2} \times \frac{n+3}{3} \times \frac{n+4}{4}$, &c. and will be = the reſpective figurate Numbers of the Order, whoſe ſide is m (by the Nature of Polygonals.)

$= 1, 1 \times m, 1 \times \frac{m}{1} \times \frac{m+1}{2}, 1 \times \frac{m}{1} \times \frac{m+1}{2} \times \frac{m+2}{3}, 1 \times \frac{m}{1} \times \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{m+3}{4}$, &c. reſpectively, (ſince $m = n+1$.) *Q. E. D.*

SCHOLIUM.

It is evident, that if n be = the ſide of a figurate Number of the firſt Order, or that of Units, then n will be = 1, and $1 \times n, 1 \times \frac{n}{1} \times \frac{n+1}{2}, 1 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, &c. will be = $1, 1 \times 1, 1 \times 1 \times 1, 1 \times 1 \times 1 \times 1$, &c. reſpectively, that is, each figurat Number of that Order will be = 1 which are manifeſtly the reſpective figurat Numbers of that Order: Therefore (by the preceding *Lemma*) $1, 1 \times m, 1 \times \frac{m}{1} \times \frac{m+1}{2}, 1 \times \frac{m}{1} \times \frac{m+1}{2} \times \frac{m+2}{3}$, &c. ($m$ being always = $n+1$)

$n+1$) are the reſpective figurate Numbers of the 2d Order, or that of Laterals: And if n be = the ſide of a figurate Number of the 2d Order, then, ſince 1, $1 \times n$, $1 \times n \times \frac{n+1}{2}$, $1 \times n \times \frac{n+1}{2} \times \frac{n+2}{3}$, &c. are (by this *Scholium*) = the reſpective figurate Numbers of the 2d Order, 1, $1 \times m$, $1 \times \frac{m}{1} \times \frac{m+1}{2}$, $1 \times \frac{m}{1} \times \frac{m+1}{2} \times \frac{m+2}{3}$, &c. will be (by the preceding *Lemma*) = the reſpective figurate Numbers of the 3d Order: Again, If n be = 3, the ſide of a figurat Number of the 3d Order, then (by the preceeding *Lemma*, ſince 1, $1 \times n$, $1 \times n \times \frac{n+1}{2}$, $1 \times n \times \frac{n+1}{2} \times \frac{n+2}{3}$, &c. are the reſpective figurate Numbers of the 3d Order) 1, $1 \times m$, $1 \times \frac{m+0}{1} \times \frac{m+1}{2}$, $1 \times \frac{m}{1} \times \frac{m+1}{2} \times \frac{m+2}{3}$, &c. will be = the reſpective figurate Numbers of the 4th Order; and ſo on: Whence may be deduc'd the following

COROLLARY I.

If n be = the ſide of figurate Number of any Order, then 1, $1 \times n$, $1 \times n \times \frac{n+1}{2}$, $1 \times n \times \frac{n+1}{2} \times \frac{n+2}{3}$, $1 \times \frac{n+0}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, &c. will be = 1ſt, 2d, 3d, 4th, 5th, &c. figurate Numbers of that Order reſpectively; *which is the univerſal Theorem I promis'd to give.*

COROLLARY II.

Since n being ſuppoſs'd = the ſide of any figurat Number, 1, $1 \times n$, $1 \times \frac{n}{1} \times \frac{n+1}{2}$, $1 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, &c. are = to the 1ſt, 2d, 3d, 4th, &c. figurate Numbers of that Order, whoſe ſide is n reſpectively; therefore the following n being ſuppos'd = number of Terms, 1, $1 \times n$, $1 \times n \times$

$1 \times n \times \frac{n+1}{2}$, $1 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, &c. ſhall be (by *Obſervation* 2. and 1ſt *Corollary*) = the nth figurate Number of the 1ſt, 2d, 3d, 4th, &c. Order reſpectively.

SCHOLIUM.

Hence alſo may be had the Sum of the Powers of an $\div$, whoſe firſt Term and common Exceſs are each = 1; thus, Suppoſing n = number of Terms of any Order, $1 \times n \times \frac{n+1}{2} = \frac{n^2+n}{2}$, will be (by *Obſ.* 1. and 2d *Coroll.*) = Sum of the 2d Series, or of the Series of Laterals to the nth Term incluſive, or = nth figurate Number of the 3d Order; therefore (by *Obſ.* 1.) $\frac{\overline{1^2+1}+\overline{2^2+2}+\overline{3^2+3}+\overline{4^2+4}+\&c.}{2}$ to n Places continued, (the Figures under each Line of Conjunction being what is here called a Place) is = the Sum of the figurate Numbers of the 3d Order, to the nth Term incluſive, or = the nth figurat Number of the 4th Order = $1 \times n \times \frac{n+1}{2} \times \frac{n+2}{3}$ (by *Coroll.* 1 & 2.) = $\frac{n^3+3n^2+2n}{6}$; But $\frac{1+2+3+4+\&c.}{2}$ to n Places, or Terms continued, is (by what has been already ſaid) = $\frac{n^2+n}{2 \times 2}$ = $\frac{1\frac{1}{2}n^2+1\frac{1}{2}n}{6}$; wherefore $\frac{1^2+2^2+3^2+4^2+\&c.}{2}$ to n Terms continued, is = $\frac{n^3+3n^2+2n}{6} - \frac{1\frac{1}{2}n^2+1\frac{1}{2}n}{6}$ = $\frac{n^3+\frac{3}{2}nn+\frac{1}{2}n}{6}$; conſequently $1^2+2^2+3^2+4^2+5^2+\&c.$ to n Terms continued, or the Sum of the Squares of an $\div$, whoſe 1ſt Term and common Exceſs are each equal to 1, and number of Terms equal to n is = $\frac{n^3+1\frac{1}{2}n^2+\frac{1}{2}n}{3}$.

Again, Since $\frac{n^3+3n^2+2n}{6}$ is = *n*th figurate Number of the 4th Order; therefore

$$\frac{\overline{1^3+3\times1^2+2\times1}+\overline{2^3+3\times2^2+2\times2}+\overline{3^3+3\times3^2+2\times3}+\overline{4^3+3\times4^2+2\times4}+\&c.\text{ to } n \text{ Places continued}}{6}$$

, is (by *Obſ.* 1.) = to the Sum of the figurate Numbers of the 4th Order to *n* Places continued, or = *n*th figurate Number of the 5th Order, which is (by the two foregoing *Corollaries*) $= 1\times\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}=\frac{n^4+6n^3+11n^2+6n}{24}$.

But (by what has been ſaid in this *Scholium*)

$$\frac{3\times\overline{1^2+2^2+3^2+4^2+\&c.\text{ to } n \text{ Terms continued}}}{6}$$

Is $=\frac{3\times\overline{n^3+1\frac{1}{2}nn+\frac{1}{2}n}}{6\times3}=\frac{4n^3+6n^2+2n}{24}$:

And $\frac{2\times\overline{1+2+3+4+\&c.\text{ to } n \text{ Terms continued}}}{6}$

$=\frac{2\times\overline{nn+n}}{6\times2}=\frac{4nn+4n}{24}$; wherefore

$\frac{1^3+2^3+3^3+4^3+\&c.\text{ to } n \text{ Terms continued}}{6}$, is =

$\frac{n^4+6n^3+11n^2+6n}{24}-\frac{4n^3+6n^2+2n}{24}-\frac{4n^2+4n}{24}=\frac{n^4+2n^3+n^2}{24}$; conſequently $1^3+2^3+3^3+4^3+5^3+$ &c. to *n* Terms continued, or the Sum of the Cubes of an ÷ whoſe firſt Term and common Exceſs are each equal to 1, and number of Terms equal to *n* is $=\frac{n^4+2n^3+n^2}{4}$.

And ſince $\frac{n^4+6n^3+11n^2+6n}{24}$ is = the *n*th figurate Number of the 5th Order, it is evident (by *Obſ.* 1.) that

$$\frac{\overline{1^4+6\times1^3+11\times1^2+6\times1}+\overline{2^4+6\times2^3+11\times2^2+6\times2}}{24}$$

$$\frac{\overline{3^4+6\times3^3+11\times3^2+6\times3}+\overline{4^4+6\times4^3+11\times4^2+6\times4}}{}$$

$\frac{+\text{ \&c. continued to } n \text{ Places}}{}$, is = to the Sum of the figurate Numbers of the 5th Order to *n* Places continued, = to the *n*th figurate Number of the 6th Order, which is (by the two foregoing *Corollaries*) $= 1\times\frac{n}{1}\times\frac{n+1}{2}\times$ $\frac{n+2}{3}\times\frac{n+3}{4}\times\frac{n+4}{5}=\frac{n^5+10n^4+35n^3+50n^2+24n}{120}$:

But (by what has been already ſaid in this *Scholium*) 'tis manifeſt that $\frac{6\times\overline{1^3+2^3+3^3+4^3+\text{ \&c. to } n \text{ Terms}}}{24}$ $\frac{\text{continued}}{}=\frac{6\times\overline{n^4+2n^3+nn}}{24\times4}=\frac{7\frac{1}{2}n^4+15n^3+7\frac{1}{2}n^2}{120}$;

Alſo $\frac{11\times\overline{1^2+2^2+3^2+4^2+\text{\&c. to } n \text{ Terms con-}}}{24}$ $\frac{\text{tinued,}}{}=\frac{11\times\overline{n^3+\frac{3}{2}n^2+\frac{1}{2}n}}{24\times3}=\frac{18\frac{1}{3}n^3+27\frac{1}{2}n^2+9\frac{1}{6}n}{120}$;

and $\frac{6\times\overline{1+2+3+4,\text{ \&c. to } n \text{ Terms continued}}}{24}$, = $\frac{6\times\overline{n^2+n}}{24\times2}=\frac{15n^2+15n}{120}$; wherefore $\frac{1^4+2^4+3^4}{24}$ $\frac{+4^4+\text{ \&c. to } n \text{ Terms continued,}}{}$ is $=\frac{n^5+10n^4+}{120}$

$$\frac{35n^3+50n^2+24n - 7\frac{1}{2}n^4+15n^3+7\frac{1}{2}nn}{120} -$$

$$\frac{18\frac{1}{3}n^3+27\frac{1}{2}nn+9\frac{1}{6}n}{120} - \frac{15nn+15n}{120} =$$

$\frac{n^5+2\frac{1}{2}n^4+1\frac{2}{3}n^3 * -\frac{1}{6}n}{120}$; conſequently $1^4+2^4+3^4+4^4+$ &c. to n Terms continued, or the Sum of the Biquadrats of an $\div$, whoſe firſt Term and common Exceſs are each equal to 1, and number of Terms equal to n is $=\frac{n^5+2\frac{1}{2}n^4+1\frac{2}{3}n^3-\frac{1}{6}n}{5}$.

And laſtly, If p be $=$ any finite Affirmative whole Number > 1, and $n =$ an indefinite Number; then it is evident, from what has been already ſaid, that the nth figurate Number of the $\overline{p+1}$th Order is $= \frac{n^p \ominus}{p\times\overline{p-1}\times\overline{p-2}\times\&c.\times 1}$ and the Coefficient of n^{p-1}; that is of the 2d Member of the Numerator of the Value of the ſaid nth figurate Number of the $\overline{p+1}$th Order is $=$ * 3d figurate Number of the $\overline{p-1}$th Order $=$ (by *Corollaries* 1ſt and 2d) $1\times\frac{p-1}{1}\times\frac{p}{2}=\frac{p^2-p}{2}$; Therefore

* *See the Table.*

$$\frac{n^p+\frac{\overline{pp-p}}{2}\times n^{p-1}\ominus}{p\times\overline{p-1}\times\overline{p-2}\times\overline{p-3}\times\&c.\times 1}$$ is $=$ the nth figurate Number of the $\overline{p+1}$th Order; therefore (by *Obſervation I.*)

$$\frac{1^p+2^p+3^p+4^p+\&c.\text{ to } n \text{ Terms continued}}{p\times\overline{p-1}\times\overline{p-2}\times\overline{p-3}\times\&c.\times 1} +$$

$$\frac{\frac{\overline{pp-p}}{2}\times\overline{1^{p-1}+2^{p-1}+3^{p-1}+4^{p-1}+\&c.}\text{ to } n \text{ Terms}}{p\times\overline{p-1}\times\overline{p-2}\times\overline{p-3}\times\&c.\text{ to } p \text{ Terms}}$$

continued

$\ominus$ is

Θ is = Sum of the figurat Numbers of the $\overline{p+1}$th Order to n Terms continued, or = nth figurate Number of the $\overline{p+2}$th Order, which is (by *Corollary* the firſt and ſecond)

$$= \frac{\overline{n^p + \frac{\overline{pp-p}}{2} \times n^{p-1}\,\Theta}}{p \times \overline{p-1} \times \overline{p-2} \times \overline{p-3} \times \&c. \times 1} \times \frac{\overline{n+p}}{p+1} =$$

$$\frac{n^{p+1} + \frac{\overline{pp+p}}{2} \times n^p\,\Theta}{\overline{p+1} \times p \times \overline{p-1} \times \overline{p-2}\ \&c. \times 1};$$ But $\frac{\overline{pp-p}}{2} \times \frac{\overline{1^{p-1} + 2^{p-1} + 3^{p-1} + 4^{p-1} + \&c. \text{ to } n}}{p \times \overline{p-1} \times \overline{p-2} \times \overline{p-3} \times \&c. \text{ to } p}$ Terms continued, is (by this *Scholium*) $= \frac{\overline{pp-p}}{2} \times \frac{n}{p}\,\Theta \div \overline{p \times \overline{p-1} \times \overline{p-2} \times \overline{p-3} \times \&c. \times 1}$

$$+ \frac{\frac{\overline{pp-1}}{2} \times n^p\,\Theta}{\overline{p+1} \times p \times \overline{p-1} \times \overline{p-2} \times \overline{p-3} \times \&c. \text{ to } \overline{p+1} \text{ Terms cond.}}$$

wherefore $\frac{1^p + 2^p + 3^p + 4^p + \&c. \text{ to } n \text{ Terms continued}}{p \times \overline{p-1} \times \overline{p-2} \times \overline{p-3} \times \&c. \times 1}$

$$= \frac{n^{p+1} + \overline{\tfrac{1}{2}pp + \tfrac{1}{2}p} \times n^p\,\Theta - \overline{\tfrac{1}{2}pp - \tfrac{1}{2}} \times n^p\,\Theta}{\overline{p+1} \times p \times \overline{p-1} \times \overline{p-2} \times \overline{p-3} \times \&c. \times 1}$$

$$= \frac{n^{p+1} + \overline{\tfrac{1}{2}p + \tfrac{1}{2}} \times n^p\,\Phi}{\overline{p+1} \times p \times \overline{p-1} \times \overline{p-2} \times \overline{p-3} \times \&c. \times 1};$$

conſequently $1^p + 2^p + 3^p + 4^p +$ &c. continued to n Terms is $= \frac{n^{p+1} + \overline{\tfrac{1}{2}p + \tfrac{1}{2}} \times n^p\,\Phi}{p+1} = \frac{n^{p+1}}{p+1} + \tfrac{1}{2} \times n^p\,\Phi.$ Canon.

N. B. Altho' the 2d Term of this Canon, namely $\tfrac{1}{2} \times n^p$ be indefinitely leſs than the firſt, *viz.* $\frac{n^{p+1}}{p+1}$, yet it will be convenient to have it inſerted, for the great Uſe we have of the ſaid Canon (which will appear in *Book II.*): And ſince the 3d Term is indefinitely leſs than the 2d, and the 4th Term indefinitely leſs than the 3d, &c. it will be needleſs to include either the 3d, 4th, &c. Terms in it, only by ⊖, and therefore the ſaid Canon will ſtand after the moſt convenient manner for our Uſe, as we have already deſign'd it.

PART XV.

Of the Nature of Series, *or of* Approximations.

CHAP. I.

The Demonſtration of Sir *Iſaac Newton's* Theorem.

LEMMA.

IF n be = ſuch a Number, as that 1, n, $\frac{n}{1}\times\frac{n-1}{2}$, $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$, $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$, &c. be = the Unciæ of the nth Power of a Binomial or Reſidual, and if m be $= n+1$; I ſay that 1, m, $\frac{m}{1}\times\frac{m-1}{2}$, $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}$, $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}$, $\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}$, &c. ſhall be = the Unciæ of the mth Power of the ſaid Binomial or Reſidual.

DEMONSTRATION.

It is evident by the Geneſis of Powers, If $1, n, \frac{n}{1}\times\frac{n-1}{2}, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$, &c. be = the Unciæ of the nth Power of a Binomial, or Reſidual, that the Unciæ of the $\overline{n+1}$ th Power of a Binomial or Reſidual will be $= 1, n+1, \frac{n}{1}\times\frac{n-1}{2}+n, n\times\frac{n-1}{2}\times\frac{n-2}{3}+\frac{n}{1}\times\frac{n-1}{2}, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}+\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$, &c.

$= 1, n+1, \frac{n}{1}\times\overline{\frac{n-1}{2}+1}\left(\frac{n+1}{2}\right), \frac{n}{1}\times\frac{n-1}{2}\times\overline{\frac{n-2}{3}+1}\left(\frac{n+1}{3}\right), \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\overline{\frac{n-3}{4}+1}\left(\frac{n+1}{4}\right)$, &c. reſpectively (by the nature of Multiplication.)

$= 1, n+1, \frac{n+1}{1}\times\frac{n}{2}, \frac{n+1}{1}\times\frac{n}{2}\times\frac{n-1}{3}, \frac{n+1}{1}\times\frac{n}{2}\times\frac{n-1}{3}\times\frac{n-2}{4}$, &c. reſpectively (ſince a, e, z and y being equal to any four Quantities $\frac{a}{e}\times\frac{z}{y}$ is $=\frac{z}{e}\times\frac{a}{y}$)

$= 1, m, \frac{m}{1}\times\frac{m-1}{2}, \frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}, \frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}$, &c. reſpectively (ſince m is $=$ $n+1$).

And that they will be ſo *in Infinitum*, ſufficiently appears the nature of the Operation.

SCHOLIA

SCHOLIA.

1. If n be $= 1$, the Index of the Root, and $m = n + 1 = 2$ the Index of the Square: Then ſince 1, $n\ (= 1)$, $\frac{n}{1} \times \frac{n-1}{2}\ (= 0)$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}\ (= 0)$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}\ (= 0)$, &c. are = the Unciæ of the Root of a Binomial or Reſidual, $1, m, \frac{m}{1} \times \frac{m-1}{2}$, ($\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, &c.) will be (by our *Lemma*) = the Unciæ of the Square of a Binomial or Reſidual.

Again, If n be $= 2$, the Index of the Square, and $m = n + 1 = 3$ the Index of the Cube; then ſince 1, $n\ (= 2)$ $\frac{n}{1} \times \frac{n-1}{2}\ (= 1)$, ($\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}\ (= 0)$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}\ (= 0)$, &c.) are = the Unciæ of the Square of a Binomial or Reſidual, $1, m, \frac{m}{1} \times \frac{m-1}{2}$ $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, ($\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$ &c.) will be (by our *Lemma*) = the Unciæ of the Cube of a Binomial or Reſidual.

Again, ſuppoſing $n = 3$ the Index of the Cube; then, ſince (by what has been already ſaid) $1, n, \frac{n}{1} \times \frac{n-1}{2}$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$, ($\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c.) are = the Unciæ of the Cube of a Binomial or Reſidual, $1, m, \frac{m}{1} \times \frac{m-1}{2}$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$, $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$

$\frac{m-2}{3} \times \frac{m-3}{4}$, (&c.) will be (by our *Lemma*) = the Unciæ of the 4th Power of a Binomial, or Reſidual, &c. Hence the *n*th Power of any Binomial or Reſidual $a \pm x$ (*n* being = any Affirmative whole Number) is $= a^n + n a^{n-1} \times \pm x + \frac{n}{1} \times \frac{n-1}{2} a^{n-2} \times + x^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} \times \pm x^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4} x^4$, &c.

Again, If *n* and *m* be = any Affirmative whole Numbers, then $\overline{a \pm x}|^n$ is (by what has been before ſaid) $= a^n \pm n a^{n-1} x + \frac{n}{1} \times \frac{n-1}{2} a^{n-2} x^2$, &c. and $\overline{a \pm x}|^m = a^m \pm m a^{m-1} x + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} x^2$, &c. and $\overline{a \pm x}|^n \times \overline{a \pm x}|^m = \overline{a \pm x}|^{m+n} = a^{m+n} \pm \overline{m+n}\, a^{m+n-1} x + \overline{m+n} \times \frac{m+n-1}{2} a^{m+n-2} x^2$, &c. Whence

$$a^m \pm m a^{m-1} x + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} x^2, \text{ \&c.}$$

$$\times\ a^n \pm n a^{n-1} x + \frac{n}{1} \times \frac{n-1}{2} a^{n-2} x^2, \text{ \&c.}$$

Produces $a^{m+n} \pm \overline{m+n}\, a^{m+n-1} x + \frac{m+n}{1} \times \frac{m+n-1}{2} a^{m+n-2} x^2$, &c.

The Truth of this will appear from Algebraick Multiplication, as far as you are pleas'd to continue the Operation; and that it will be ſo *ad infinitum*, is (I think) manifeſt from what has been already ſaid.

* Now ſince this product is ſuch as I have expreſs'd, it muſt be ſo, altho' *n* and *m* were each, or both $=$ any Numbers whatſoever; for this Multiplication does not diſtinguiſh what Numbers they are equal to, but on the contrary, being Symbols, are to be conſider'd therein only as ſuch, that is, as Univerſal. Hence

2. $\overline{a \pm x}|^{\frac{1}{n}}$ is $= a^{\frac{1}{n}} \pm \frac{1}{n} a^{\frac{1}{n}-1} x + \frac{\frac{1}{n}}{1} \times \frac{\frac{1}{n}-1}{2} a^{\frac{1}{n}-2} x^2 \pm \frac{\frac{1}{n}}{1} \times \frac{\frac{1}{n}-1}{2} \times \frac{\frac{1}{n}-2}{3} a^{\frac{1}{n}-3} x^3$, &c. *In infinitum*; for this multiplyed by it ſelf is $(= \overline{a \pm x}|^{\frac{2}{n}}) = a^{\frac{2}{n}} \pm \frac{2}{n} a^{\frac{2}{n}-1} x + \frac{\frac{2}{n}}{1} \times \frac{\frac{2}{n}-1}{2} a^{\frac{2}{n}-2} x^2$, &c. (by what was before ſaid), and this multiplied by $\overline{a \pm x}|^{\frac{1}{n}}$ is $(= \overline{a \pm x}|^{\frac{3}{n}})$ $= a^{\frac{3}{n}} \pm \frac{3}{n} a^{\frac{3}{n}-1} x + \frac{\frac{3}{n}}{n} \times \frac{\frac{3}{n}-1}{2} a^{\frac{3}{n}-2} x^2$, &c. & therefore $\overline{a^{\frac{1}{n}} \pm \frac{1}{n} a^{\frac{1}{n}-1} x + \frac{\frac{1}{n}}{1} \times \frac{\frac{1}{n}-1}{2} a^{\frac{1}{n}-2} x^2, \&c.}|^{n}$ is $= a^{\frac{n}{n}} \pm \frac{n}{n} a^{\frac{n}{n}-1} x + \frac{\frac{n}{n}}{1} \times \frac{\frac{n}{n}-1}{2} a^{\frac{n}{n}-2} x^2$, &c. that is $= a \pm x$.

Hence it is manifeſt that $\overline{a \pm x}|^{\frac{m}{n}}$ (*m* and *n* being $=$ affirmative Numbers) is $= a^{\frac{m}{n}} \pm \frac{m}{n} a^{\frac{m}{n}-1} x + \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n}-1}{2} a^{\frac{m}{n}-2} xx$, &c. Wherefore

3. $\overline{a \pm x}|^{-\frac{m}{n}}$ is $= a^{-\frac{m}{n}} \pm \frac{-m}{n} a^{-\frac{m}{n}-1} x + \frac{-\frac{m}{n}}{1} \times \frac{-\frac{m}{n}-1}{2} a^{-\frac{m}{n}-2} x^2$, &c. For this multiplied

by $\overline{a \pm x}|^{\frac{m}{n}}$ when evolved as above, is (by what was before ſaid) $= a^{\frac{m}{n} - \frac{m}{n}} \pm \overline{\frac{m}{n} - \frac{m}{n}} \times a^{\frac{m}{n} - \frac{m}{n} - 1} x$; &c. $=$ $a^{\frac{m}{n} - \frac{m}{n}} + 0 = 1$ $(= \overline{a \pm x}|^{\frac{m}{n} - \frac{m}{n}})$.

COROLLARY.

If any Binomial or Reſidual $a \pm x$ be raiſed to any Power; whoſe Index is n (n repreſenting any Number whatſoever) the nth Power of that Binomial, or Reſidual, will be $=$

$$a^n + n a^{n-1} \times \pm x + \frac{n}{1} \times \frac{n-1}{2} a^{n-2} x^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} \times \pm x^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4} \times + x^4 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} a^{n-5} \times \pm x^5, \text{ \&c.}$$

Or putting $q = \frac{x}{a}$, then $a \pm x = a \pm a \times \frac{x}{a} = a \pm a \times q$ $(= a \times \overline{1 \pm q})$; and $\overline{a \pm a q}|^n$ is $= a^n \pm n a^n q + \frac{n}{1} \times \frac{n-1}{2} d^n q^2 \pm \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^n q^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^n q^4$, &c.

Or putting A $=$ 1ſt Term, B $=$ 2d, C $=$ 3d, D $=$ 4th, &c. then it will be $\overline{a \pm a q}|^n = a^n + n \text{A} \times \pm q + \frac{n-1}{2} \text{B} \times \pm q + \frac{n-2}{3} \text{C} \times \pm q + \frac{n-3}{4} \text{D} \times \pm q$, &c.

Or if it were required to raiſe a Trinomial, Quadrinomial, &c. or Infinitomial to any given Power, it may be done by the foregoing *Corollary*: As for Inſtance, If it were required to raiſe the Trinomial $a+b+c$ to the nth Power: It may be done thus;

$$\overline{a+b+c}|^{n} = a^{n} + na^{n-1} \times \overline{b+c} + \frac{n}{1} \times \frac{n-1}{2}$$

$$a^{n-2} \times \overline{b+c}|^{2} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3} \times \overline{b+c}|^{3} +$$

&c. Or,

Suppoſe the Infinitomial $az + bz^{2} + cz^{3} + dz^{4} + \&c.$ were to be rais'd to the Power whoſe Index is n; then,

$$\overline{az + bz^{2} + cz^{3} + dz^{4} + \&c.\ \textit{Sine fine}}|^{n} =$$

$$a^{n}z^{n} + n\overline{az}|^{n-1} \times \overline{bz^{2} + cz^{3} + dz^{4} + \&c.\ \textit{Sine fine}}$$

$$+ \frac{n}{1} \times \frac{n-1}{2} \times \overline{az}|^{n-2} \times \overline{bz^{2} + cz^{3} + dz^{4} + \&c.}$$

$$\overline{\textit{Sine fine}}|^{2} + \&c.$$

CHAP. II.

Of the CONVERGING SERIES.

See Pages 65 and 66. MR. *Ralphſon's* Method in raiſing particular Theorems for Extracting the Roots of all Æquations whatſoever, is as follows, *viz.* He ſuppoſes the Root in any Æquation, which he calls a, to be = a Binomial of g a known, and x an unknown Quantity; and then ſubſtitutes the ſaid Binomial $g+x$, and its reſpective Powers for a and its Powers in the given Æquation, and then rejects (or leaves out) all the Members of the new Æquation, wherein there is any Power of x (exceeding the Root) and with the other Members he makes a ſuppos'd Æquation, and thereby finds the ſuppos'd Value of x, or the firſt Theorem required. But if this Method be applyed to an Univerſal Æquation, an Univerſal Theorem will thereby be rais'd, which will include all his firſt ſett of Theorems, and indeed all others of that kind, for extracting the Roots of all Æquations whatſoever: Let us therefore ſuppoſe the given Æquation to be

$$a^{n} \pm pa^{n-1} \pm qa^{n-2} \pm ra^{n-3} \pm \&c. = \pm N.$$

In which N is = the given abſolute Number.

n = Exponent of the higheſt Power of a the Root ſought.

N. B. *n muſt be not leſs* 2.

a = Root ſought.

$1, p, q, r,$ &c. = reſpective Coefficients of the Members of the given Æquation.

Let us alſo ſuppoſe $g + x = a$, which g is the known Part of the Root ſought a, and ought to be taken as near the ſought Root as may be, whether it be Greater or Leſs than the ſaid Root, and x is = the unknown Part of the Root ſought, whoſe Value may be Negative or Affirmative, according as that of g is taken Greater or Leſs than juſt.

Then $\overline{g+x}|^{n} \pm p \times \overline{g+x}|^{n-1} \pm q \times \overline{g+x}|^{n-2} \pm r \times \overline{g+x}|^{n-3} \pm$ &c. $= \pm$ N.

That is, by Sir *Iſaac Newton*'s Theorem,

$$\left.\begin{array}{l}
\overline{g+x}|^{n} = \left\{\begin{array}{c} g^{n} + ng^{n-1}x + \\ n \times \frac{n-1}{2} g^{n-2} x^{2} + \&c. \end{array}\right\} \\[2ex]
\pm p \times \overline{g+x}|^{n-1} = \left\{\begin{array}{l} \overline{\pm p \times g^{n-1} + \overline{n-1}\, g^{n-2} x} \\ + \frac{n-1}{1} \times \frac{n-2}{2} g^{n-3} x^{2} + \&c. \end{array}\right\} \\[2ex]
\pm q \times \overline{g+x}|^{n-2} = \left\{\begin{array}{l} \overline{\pm q \times g^{n-2} + \overline{n-2}\, g^{n-3} x} \\ + \frac{n-2}{1} \times \frac{n-3}{2} g^{n-4} x^{2} + \&c. \end{array}\right\} \\[2ex]
\pm r \times \overline{g+x}|^{n-3} = \left\{\begin{array}{l} \overline{\pm r \times g^{n-3} + \overline{n-3}\, g^{n-4} x} \\ + \frac{n-3}{1} \times \frac{n-4}{2} g^{n-5} x^{2} \pm \&c. \end{array}\right\} \\[2ex]
\&c. = \&c.
\end{array}\right\} = \pm \text{N}$$

Now, by rejecting all the Members wherein any Power of x, exceeding the Root, is contain'd; we have

$$\left.\begin{array}{ll} g^n & + n g^{n-1} x \\ \pm p g^{n-1} & \pm \overline{pn - p}\, g^{n-2} x \\ \pm q g^{n-2} & \pm \overline{qn - 2q}\, g^{n-3} x \\ \pm r g^{n-3} & \pm \overline{rn - 3r}\, g^{n-4} x \\ & \text{\&c.} \end{array}\right\} = \pm N.$$

And by Tranſpoſition we have

$$n g^{n-1} x \pm \overline{pn - p}\, g^{n-2} x \pm \overline{qn - 2q}\, g^{n-3} x \pm \overline{rn - 3r}\, g^{n-4} x \pm \text{\&c.} = \pm N - g^n \mp p g^{n-1} \mp q g^{n-2} \mp r g^{n-3} \mp \text{\&c.}$$

And by Diviſion we have

$$x = \frac{\pm N - g^n \mp p g^{n-1} \mp q g^{n-2} \mp r g^{n-3} \mp \text{\&c.}}{n g^{n-1} \pm \overline{pn - p}\, g^{n-2} \pm \overline{qn - 2q}\, g^{n-3} \pm \overline{rn - 3r}\, g^{n-4} \pm \text{\&c.}}$$

which Theorem exhibits all poſſible Particular ones for extracting Roots, according to the firſt ſort of Mr. *Raphſon's* agreeing exactly with them, as will be found on Tryal; always remembring that the Signs in the Dividend muſt be contrary to thoſe in the Æquation, and in the Diviſor the ſame reſpectively: And likewiſe it will be proper to take Notice, that if any Term be wanting in the Æquation, the ſame muſt be omitted in the *Theorem*. Thus,

If $a^n = N$, then $\left\{\begin{array}{l} p = 0 \\ q = 0 \\ r = 0 \\ \text{\&c.} \end{array}\right\}$ and

$x = \dfrac{N - g^n}{n g^{n-1}}$ which is an Univerſal Canon or Theorem for evolving the Roots of all Pure Powers whatſoever.

Now the Method of finding the Affirmative Roots of any Æquation whatſoever, by the above univerſal Theorem is this.

1ſt. Deduce from the ſaid univerſal Theorem the Particular one for extracting the Root (or Roots) of the given Æquation; then ſuppoſe $g =$ ſome Number, as near to the required Value of a, as you can gueſs; and by the ſaid particular Theorem, find the ſuppos'd Value of x, which add to the ſaid ſuppos'd Value of g, and call that Sum g, that is your 2d g; with which 2d g (by the ſaid particular Theorem) find another x, and call the Sum of the laſt found x, and of g the 2d, your 3d g; and ſo proceed till ſome g be $= a$, the true Root required, or as near it as is ſufficient.

Now that this manner of proceeding will give the true Value of *a Proxime* in the foregoing univerſal ſimple Æquation, I demonſtrate thus:

The ſuppos'd Value of g muſt be either $=$ to the * Affirmative Value a, or leſs than it, or greater.

Caſe 1. If it be equal to it (that is, if $g = a$) then the Theorem, which we are about Demonſtrating, to wit, $x = \frac{N - g^n}{n g^{n-1}}$ will become $x = \frac{N - a^n}{n a^{n-1}}$; that is to ſay, $x = 0$, as it muſt, in that Caſe, evidently be.

2dly. If the Value of g be taken Leſs than that of a; then, I ſay, the Value of the 2d g will be Greater than that of a:

$$\text{For 2d } g = g + \frac{N - g^n}{n g^{n-1}}$$

$$\text{And } a = g + \frac{\overline{N - g^n - \frac{n}{1} \times \frac{n-1}{2} g^{n-2} x^2, \text{ \&c.}}}{n g^{n-1}}$$

But ſince the Value of g (which is ſuppoſed to be Affirmative) is Leſs than that of a; the true Value of x will be Affirmative; conſequently (n being ſuppos'd not $-\!\sqsupset 2$)

$$\frac{N - g^n}{n g^{n-1}} \sqsubset \frac{\overline{N - g^n - n \times \frac{n-1}{2} g^{n-2} x^2 + \text{\&c.}}}{n g^{n-1}}$$

---

* *For it is manifeſt it hath but one Value that is Affirmative, which is what we ſeek.*

therefore $g + \frac{N - g^n}{ng^{n-1}}$ $(= 2\text{d } g) \sqsubset$

$$g + \frac{N - g^n - n \times \frac{n-1}{2} g^{n-2} x^2 + \&c.}{ng^{n-1}} \quad (= a).$$

3*dly.* If the Value of any g be greater than that of a; I ſay the Value of the next following g will ſtill be $\sqsubset a$; but $\sqsupset$ the foregoing g: for

$$\text{Following } g = g + \frac{N - g^n}{ng^{n-1}}$$

$$\text{And } a = g + \frac{N - g^n - \frac{n}{1} \times \frac{n-1}{2} g^{n-2} x^2 + n \times \frac{n-1}{2}}{ng^{n-1}}$$

$$\overline{\times \frac{n-2}{3} g^{n-3} x^3 + \&c.}$$

But ſince g is $\sqsubset a$, the true Value of x is Negative: Let us therefore deſign it here by a Negative Symbol, *viz.* $-z$; then it is evident that

$$g - z \sqsubset g - z - \frac{n-1}{2} g^{-1} z^2 - {*}\ \&c. \quad \textit{Sine fine.}$$

That is $g - z$ is $\sqsubset \overline{g^n - ng^{n-1} z}|^{\frac{1}{n}}$; wherefore

$$\overline{g - z}|^n = g^n - ng^{n-1} z + n \times \frac{n-1}{2} g^{n-2} z^2 - n \times \frac{n-1}{2} \times \frac{n-2}{3} g^{n-3} z^3, \&c. \text{ is } \sqsubset g^n - ng^{n-1} z:$$

$$\text{whence } \frac{n}{1} \times \frac{n-1}{2} g^{n-2} z^2 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} g^{n-3} z^3, \&c. \text{ is } \sqsubset 0.$$

---

* *For* $\frac{\frac{1}{n} - 2}{3}$, *or* $\frac{\frac{1}{n} - 3}{4}$, or $\frac{\frac{1}{n} - 4}{5}$, *or &c. is = Negative, which × Affirmative ×* $-z$ *produces an Affirmative; and this × next preceding Term, which is Negative, will produce a Negative.*

And by ſubſtituting $+x$ for $-z$, we have $\frac{n}{1}\times\frac{n-1}{2}g^{n-2}x^2 + {}^{*}\,\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}g^{n-3}x^3 + \&c \sqsubset 0$.

Conſequently $\frac{N-g^n}{ng^{n-1}} \sqsubset \frac{N-g^n-\frac{n}{1}\times\frac{n-1}{2}g^{n-2}x^2-\&c.}{ng^{n-1}}$

Therefore $g+\frac{N-g^n}{ng^{n-1}}$ (= following g) $\sqsubset$

$$g+\frac{N-g^n-\frac{n}{1}\times\frac{n-1}{2}g^{n-2}x^2-\&c.}{ng^{n-1}} (\sqsubset a).$$

That is to ſay, if the Value of the foregoing g be $\sqsubset a$, the Value of the following g. Will be alſo $\sqsubset a$.

But $\frac{N-g^n}{ng^{n-1}} \sqsupset 0$.

Therefore $g+\frac{N-g^n}{ng^{n-1}}$ (= following g) is $\sqsupset g$:

That is to ſay, each following g in leſs in Value than the foregoing g; but (by what has been before ſaid) ſtill Greater than the Value of a: Whence (ſince the 2d Caſe is reduc'd to the 3d, as it is by one only Renewal) the approaches in the 2d and 3d Caſes are demonſtrated.

From the foregoing Demonſtration it is obvious, that as x is ſmall in reſpect of g, the Series muſt Converge accordingly quick; for g wanting only x of being $= a$, the next g will want but $\frac{n-1}{2}g^{-2}x^2 + \&c.$ which, if x be but very ſmall in reſpect of g, is but inconſiderably more than $\frac{\overline{n-1}\,x^2}{2g}$. Now this Quantity muſt be by much leſs than x, if x be much leſs than g.

This

* Its Value is not $\sqsubset 0$.

This Series Converges ſo ſwift, not only in the foregoing Simple Æquations, but alſo in all Adfected Æquations when you have got ſo many of the Figures of the Root you are about Extracting, as to free it from the Intanglements and Incumbrances of the other Roots, as intirely, or almoſt to double the true Figures in the aſſumed g, by each renewal.

Having thus demonſtrated the truth of our univerſal Canon for evolving the Roots of all Pure (or Simple) Powers; I will in the next Place, endeavour to demonſtrate (as far as it can be done) our univerſal Canon for evolving the Roots of any Æquation whatſoever: Thus,

The Value of g muſt be taken either = one of the Values of a, or greater than the Greateſt, or leſs than the Leaſt of its Values, or leſs than the Greateſt, and greater than the Leaſt Value of a.

I. If the Value of g be taken = any of the Values of a; then the Theorem, which we are to Demonſtrate, being

$$x = \frac{\pm N - g^n \mp p g^{n-1} \mp q g^{n-2} \mp \&c.}{n g^{n-1} \pm p \times \frac{n-1}{1} g^{n-2} \pm q \times \frac{n-2}{1} g^{n-3} \pm \&c.}$$

will become $x = \dfrac{\pm N - a^n \mp p a^{n-1} \mp q a^{n-2} \mp \&c.}{n a^{n-1} \pm p \times \frac{n-1}{1} a^{n-2} \pm q \times \frac{n-2}{1} a^{n-3} \pm \&c.}$ That is to ſay, $x = 0$, as it muſt, in that Caſe manifeſtly be.

II. If the Value of g be taken greater than the Greateſt Value of a; and that the ſaid Greateſt Value of a is not * Imaginary, by which means it will be poſſible to extract it, at leaſt *Proxime*; then I ſay, the Value of 2d g will ſtill be ⊏ greateſt Value of a: For

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* *Thus* $4 \pm \sqrt{-1}$ *is deem'd* ⊏ 4.

2d $g =$

2d $g = g + \frac{\pm N - g^n \mp pg^{n-1} \mp qg^{n-2} \mp \&c.}{ng^{n-1} \pm p \times \frac{n-1}{1} g^{n-2} \pm q \times \frac{n-2}{1} g^{n-3} \pm \&c.}$

And each of the Values of

$$a = g + \frac{\pm N - g^n \mp pg^{n-1} \mp qg^{n-2} \mp \&c. - n \times \frac{n-1}{2} \times g^{n-2} x^2 \mp p \times \frac{n-1}{1} \times \frac{n-2}{2} g^{n-3} x^2 \mp q \times \frac{n-2}{1} \times \frac{n-3}{2} \times g^{n-4} x^2 \mp \&c. \&c.}{ng^{n-1} \pm p \times \frac{n-1}{1} g^{n-2} \pm q \times \frac{n-2}{1} \times g^{n-3} \pm \&c.}$$

Now ſince the Value of g is $\sqsubset$ greateſt Value of a, it is manifeſt that $g^n \pm pg^{n-1} \pm qg^{n-2} \pm \&c. \mp N$ will be $\sqsubset 0$: And (by 1ſt of *Corollary* to the 3 laſt Paragraphs in *Part IX.*) $ng^{n-1} \pm p \times \frac{n-1}{1} g^{n-2} \pm q \times \frac{n-2}{1} g^{n-3} \pm \&c.$ will be $\sqsubset 0$: And (by the 2d of the ſaid *Corollary*) $\underline{g^n \pm pg^{n-1} \pm qg^{n-2} \pm \&c. \mp N} + \overline{ng^{n-1} \pm p \times \frac{n-1}{1} g^{n-2} \pm q \times \frac{n-2}{1} g^{n-3} \pm \&c.} \times x$ is $\sqsupset 0$, ſince the true Values of x are each $\sqsupset 0$: And therefore the Difference of this Quantity and 0 *

$= n \times \frac{n-1}{2} g^{n-2} x^2 \pm p \times \frac{n-1}{1} \times \frac{n-2}{2} g^{n-3} x^2 \pm q \times \frac{n-2}{1} \times \frac{n-3}{2} g^{n-4} x^2 \pm \&c. \&c.$ is $\sqsubset 0$; Wherefore,

* *See Page* 122.

$$g+\frac{\pm N-g^{n}\mp pg^{n-1}\mp qg^{n-2}\mp\&c.-n\times\frac{n-1}{2}g^{n-2}x^{2}}{ng^{n-1}\pm p\times\frac{n-1}{1}g^{n-2}\pm q\times\frac{n-2}{1}g^{n-3}\pm}$$

$$\underline{\mp p\times\frac{n-1}{1}\times\frac{n-2}{2}g^{n-3}x^{2}\mp q\times\frac{n-2}{1}\times\frac{n-3}{2}g^{n-4}x^{2}\mp}$$

&c.
&c. &c.

———; that is g + the converging x — a Number greater than 0 ($\mp$ each of the Values of a) is $\sqsupset$ g +

$$\frac{\pm N-g^{n}\mp pg^{n-1}\mp qg^{n-2}\mp\&c.}{ng^{n-1}\pm p\times\frac{n-1}{1}g^{n-2}\pm q\times\frac{n-2}{1}g^{n-3}\pm\&c.}$$

that is $\sqsupset$ g + the Converging x (= 2d g).

But the Value of the Converging x

$$\left(=\frac{\pm N-g^{n}\mp pg^{n-1}\mp qg^{n-2}\mp\&c.}{ng^{n-1}\pm p\times\frac{n-1}{1}g^{n-2}\pm q\times\frac{n-2}{1}g^{n-3}\pm\&c.}\right)$$

is $\sqsupset$ 0.

Conſequently 2d g is $\sqsupset$ g (or 1ſt g).

That is to ſay, If g be taken greater than the Greateſt Value of a, then the Value of the 2d g will ſtill be $\sqsubset$ ſaid Value of a, always provided that the greateſt Value of a is not Imaginary, but $\sqsupset$ 1ſt g; whence an approach in that Caſe, towards the greateſt Value of a, is Demonſtrated.

III. If g be taken leſs than either Value of a, then the 2d g will, in ſome Æquations, be greater than the Greateſt Value of a, and in others, leſs than it; but in all Æquations greater than the 1ſt g.

IV. As to an Approach in the 4th Caſe, *viz.* when g is taken greater than one, and leſs than onother of the Values of a, it admits of Exceptions, and therefore cannot be demonſtrated.

Wherefore the ſureſt way of Extracting the Roots of Adfected Æquations, is to begin the Operation by the Numeral Exegeſis, and ſo go on with it till the Diviſor takes Place; and then purſue it by this Method, or by either of Dr. *Halley*'s Theorems, either of which will ſerve to find as many Figures of the Root you ſeek, (it being, I preſume, not Imaginary) as is requiſite.

N.B. *It will, for the moſt part, be ſufficient to find the firſt Member of the Root you ſeek, for to apply this Theorem thereto.*

From our foregoing univerſal Theorem another may be deduc'd, which will include all Mr. *Raphſon*'s 2d ſet of Theorems; thus, *viz.*

Our firſt univerſal Theorem is this,

$$x = \frac{\pm N - g^n \mp pg^{n-1} \mp qg^{n-2} \mp rg^{n-3} \mp \&c.}{ng^{n-1} \pm \overline{pn-p}\,g^{n-2} \pm \overline{qn-2q}\,g^{n-3} \pm \overline{rn-3r}\,g^{n-4} \pm \&c.}$$

Therefore $g + x$ ($=$ 2d g)

$$= \frac{\pm N + \overline{n-1}\,g^n \pm \overline{pn-2p}\,g^{n-1} \pm \overline{qn-3q}\,g^{n-2} \pm \overline{rn-4r}\,g^{n-3} \pm \&c.}{ng^{n-1} \pm \overline{pn-p}\,g^{n-2} \pm \overline{qn-2q}\,g^{n-3} \pm \overline{rn-3r}\,g^{n-4} \pm \&c.}$$

Which laſt Theorem includes all them of Mr. *Raphſon*'s 2d ſort Univerſally: But in this Theorem the Signs in the Dividend and Diviſor will be the ſame as in the given Æquation reſpectively.

Note, *That the 1ſt and 2d Univerſal Theorems are the ſame in in Effect; and therefore I will deduce only from the 1ſt of them ſome particular Theorems, and thereby extract ſome of the Roots of ſome Æquations which I will chooſe for Examples.*

$$\text{If}\left\{\begin{array}{c}\text{Æquation}\\ aa = N\\ aaa = N\\ a^4 = N\\ a^2 \pm pa = N\\ a^2 - pa = -N\\ a^3 \pm pa^2 \pm qa = N\\ a^4 \pm pa^3 = N\\ \&c.\end{array}\right\}\text{Then (by the 1ſt General Theorem)}\left\{\begin{array}{c}\frac{N - g^2}{2g}\\ \frac{N - ggg}{3gg}\\ \frac{N - gggg}{4g^3}\\ \frac{N - gg \mp pg}{2g \pm p}\\ \frac{-N - gg + pg}{2g - p}\\ \frac{N - g^3 \mp pg^2 \mp qg}{3g^2 \pm 2pg \pm q}\\ \frac{N - g^4 \mp pg^3}{4g^3 \pm 3pg^2}\\ \&c.\end{array}\right\} = x.$$

Take a few Examples of ſome of theſe particular Theorems.

Example

Example I.

If $aa = 2$; what is a equal to?

Suppoſe $g = 1$, then $x\ (= \frac{N - g^2}{2g}) = \frac{2 - 1}{2} = .5$;
Therefore $1 + .5\ (= g + x) = 2$d g.

Then x (or 2d x) $= \frac{2 - 2.25}{3} = -.083$;

Therefore $1.5 - .083 = 1.417 = 3$d g:

Then $x = \frac{2 - 2.007889}{2.834} = -.002783$; therefore $1.417 - .002783 = 1.414217 = 4$th g.

Then $x = \frac{2 - 2.000009723089}{2.828434} = -.000003437622$ therefore $1.414217 - .000003437622 = 1.414213562378$ $= g$ the 5th $= a$ ſought very near. *Anſwer.*

Example II.

If $a^4 - 4a^3 = 13824$; 'tis required to find one of the Affirmative Values of a.

Firſt I ſuppoſe $a = 10$; then $10000 - 4000\ (= a^4 - 4a^3) = 6000$; But $6000 \sqsupset 13824$; therefore $a \sqsubset 10$.

Again I ſuppoſe $a = 20$, then $160000 - 32000\ (= aaaa - 4a^3) = 128000$; But $128000 \sqsubset 13824$; therefore $a \sqsupset 20$: therefore the Value of a (or at leaſt one of them) is $\sqsupset 20$, but $\sqsubset 10$, but nearer to 10 than to 20.

Therefore ſuppoſe $g = 10$;

Then $x\ (= \frac{N - g^4 + pg^3}{4g^3 - 3pg^2}) = \frac{13824 - 10000 + 4000}{4000 - 1200}$ $= \frac{7824}{2800} = 2$; therefore $g + x = 10 + 2 = 12 = g$ the 2d $= a$ ſought.

Dr, *Edmund Halley* found two other Univerſal Theorems for Evolving Æquations, the one Rational, and the other Irrational, which, I think, are better than Mr. *Raphſon*'s Theorems; for every Renewal of each of Dr. *Halley*'s Triples the true Figures in the Value of g. The Method of finding which ſaid two Theorems is as follows.

Let us reaſſume the foregoing Univerſal Æquation, *viz.*

$$a^n \pm pa^{n-1} \pm qa^{n-2} \pm ra^{n-3} \pm \&c. = \pm N.$$

And ſuppoſe $g + x = a$;

Then $\overline{g+x}|^n \pm p\times\overline{g+x}|^{n-1} \pm q\times\overline{g+x}|^{n-2} \pm r\times\overline{g+x}|^{n-3} \pm \&c. = \pm N.$

That is, by Sir *Iſaac Newton*'s Theorem, and Tranſpoſition,

$$\left.\begin{array}{l}
g^n + ng^{n-1}x + n\times\frac{n-1}{2}g^{n-2}x^2 + \&c. \\ \hline
\pm p\times g^{n-1} + \overline{n-1}\,g^{n-2}x + \frac{n-1}{1}\times\frac{n-2}{2}g^{n-3}x^2 + \&c. \\ \hline
\pm q\times g^{n-2} + \overline{n-2}\,g^{n-3}x + \frac{n-2}{1}\times\frac{n-3}{2}g^{n-4}x^2 + \&c. \\ \hline
\pm r\times g^{n-3} + \overline{n-3}\,g^{n-4}x + \frac{n-3}{1}\times\frac{n-4}{2}g^{n-5}x^2 + \&c. \\
\qquad\qquad \&c. \mp N.
\end{array}\right\} = 0$$

Now in this Æquation for the Sum of all its Members, wherein x, or any of its Powers is not inſerted ſubſtitute $-b$; that is to ſay, let $-b$ be $= g^n \pm pg^{n-1} \pm qg^{n-2} \pm rg^{n-3} \pm \&c. \mp N.$

Alſo for the Sum of the Coefficients of the the Root x, ſubſtitute $+c$; that is to ſay, Let $+c$ be $= ng^{n-1} \pm \overline{np-p}\,g^{n-2} \pm \overline{nq-2q}\,g^{n-3} \pm \overline{rn-3r}\,g^{n-4} \pm \&c.$

Again, For the Sum of the Coefficients of xx, put d.

Alſo for the Sum of the Coefficients of x^3, x^4, x^5, &c. put $+f$, $+b$, $+k$, &c. reſpectively.

Then the foregoing Æquation will become $-b+cx+dx^2+fx^3+hx^4+$ &c. $=0$.

And, By rejecting all the Members, wherein any Power of x exceeding its Square is included, we have $-b+cx+dxx=0$.

And by Tranſpoſition,

$$cx+dx^2=b.$$

And by Diviſion,

$$x=\frac{+b}{c+dx}$$

But x, by Mr. *Raphſon's* 1ſt Sett of Theorems, (that is by our foregoing 1ſt Univerſal Theorem) is $=\frac{b}{c}$; therefore x $\left(=\frac{b}{c+dx}\right)=\frac{b}{c+d\times\frac{b}{c}}$, which is Dr. *Halley's* rational Theorem for Extracting the Roots of any Æquation whatſoever.

Again, $-b+cx+dx^2+$ &c. is (by what was before ſaid) $=0$; therefore, by rejecting all the Members of this Æquation, which includes any Power of x, exceeding xx, and afterwards tranſpoſing, we have

$$dx^2+cx=b.$$

And by Dividing each Part by d, we have

$$x^2+\frac{c}{d}x=\frac{b}{d}$$

By comp. □ $x^2+\frac{c}{d}x+\frac{cc}{4dd}=\frac{b}{d}+\frac{cc}{4dd}=\frac{bd+\frac{1}{4}cc}{dd}$.

By Evolution, $x+\frac{c}{2d}=\frac{\sqrt{bd+\frac{1}{4}cc}}{d}$.

And by Tranſpoſition,

$$x=\frac{\sqrt{bd+\frac{1}{4}cc}-\frac{1}{2}c}{d}$$

Which is Dr. *Halley*'s Irrational Theorem for Evolving all Æquations whatſoever.

Since each of Dr. Halley's *Theorems approaches nearer to the Values of* a, *than Mr.* Raphſon's, *which I have already, in ſome Meaſures, prov'd, I think it needleſs to trouble my ſelf in further proving them; I will therefore only tell you, That Dr.* Halley's *Rational Theorem is Erroneous in point of Defect, and his Irrational in Exceſs.*

By a few Examples theſe two Univerſal Theorems will be render'd more intelligible.

Example I.

If $a^3 = 2$, 'tis required to find the Value of a by Dr. *Halley*'s Theorems.

Suppoſe $g = 1$; then

$$\left.\begin{array}{l} \overline{g+x}|^3 = 1 + 3x + 3x^2 + x^3 \\ -N = -2 \end{array}\right\} = 0.$$

$$-1 + 3x + 3x^2 + x^3 = 0,$$

That is $-b + cx + dx^2 + fx^3 = 0$.

Then by Dr. *Halley*'s Rational Theorem,

$$x \left(= \frac{b}{c + \frac{bd}{c}}\right) = \frac{1}{3 + \frac{3}{3}} = \frac{1}{4} = .25.$$

And by his Irrational Theorem,

$$x \left(= \frac{\overline{bd + \frac{1}{4}cc}|^{\frac{1}{2}} - \frac{1}{2}c}{d}\right) = \frac{\overline{3 + 2\frac{1}{4}}|^{\frac{1}{2}} - \frac{3}{2}}{3} = .26.$$

Therefore $g + x$ is (by the Rational Theorem) $= 1.25$, or (by the Irrational Theorem) $= 1.26$.

Either of which values of $g + x$ may be taken for g the 2d.; But the Latter is nearer the value of a; Let us therefore ſuppoſe g (or 2d g) $= 1.26$. Then

$g +$

$$\left.\begin{array}{l}\overline{g+x}|^{3} = 2.000376 + 4.7628\,x + 3.78\,x^{2} + x^{3}\\ -N = -2\\ \hline \qquad .000376 + 4.7628\,x + 3.78\,x^{2} + x^{3}\end{array}\right\} = 0$$

That is $- \quad b \quad + \quad cx + dx^{2} + fx^{3} = 0$.

Then by the Rational Theorem,

$$x\ \left(= \frac{b}{c + \frac{bd}{c}}\right) = \frac{-.000376}{4.7628 - \frac{.000376 \times 3.78}{4.7628}}$$

$= -.000078950104$; therefore $g + x = 1.26 - .000078950104 = 1.259921049895 = a$ very near.

Or by the Irrational Theorem $x\ \left(= \frac{\overline{bd + \frac{1}{4}cc}|^{\frac{1}{2}} - \frac{1}{2}c}{d}\right) =$

$\sqrt{.396800529100529100529100529} - .63 = .629921049895 - .63$; therefore $g + x = 1.26 + .629921049895 - .63 = 1.259921049895 = a$ near enough.

Example II.

Let it be required to find one of the Affirmative Values of a in this Adfected Cubick Æquation. *Viz.*

$a^{3} + 438\,a^{2} - 7825\,a - 98508430 = 0.$

1. Suppoſe $g = 300$, then

$$\left.\begin{array}{l}\overline{g+x}|^{3} = 27000000 + 270000\,x + 900\,x^{2} + x^{3}\\ 438 \times \overline{g+x}|^{2} = 39420000 + 262800\,x + 438\,x^{2}\\ -7825 \times \overline{g+x} = -2347500 - 7825\,x\\ \qquad -N = -98508430\end{array}\right\} = 0$$

$$\overline{\qquad -34435930 + 524975\,x + 1338\,x^{2} + x^{3} = 0.}$$

That is $- \quad b \quad + \quad cx + dx^{2} + fx^{3} = 0$.

Then by the Rational Theorem

$$x\ \left(= \frac{b}{c + \frac{bd}{c}}\right) = 56;\ \text{therefore } g + x = 356.$$

Or by the Irrational Theorem,

$$x \left(= \frac{\overline{bd + \frac{1}{4}cc}\,\big|^{\frac{1}{2}} - \frac{1}{2}c}{d}\right) = 57;$$ therefore $g + x = 357$.

2. Renew the Theorem, and ſuppoſe g (or g the 2d) $= 357$, then,

$$\left.\begin{array}{r} \overline{g+x}\,|^3 = +45499293 + 382347x + 1071x^2 + x^3 \\ 438 \times \overline{g+x}\,|^2 = +55822662 + 312732x + 438x^2 \\ -7825 \times \overline{g+x} = -2793525 - 7825x \\ -N = -98508430 \end{array}\right\} = 0$$

$$+20000 + 687254x + 1509x^2 + x^3 = 0$$

That is $-b + cx + dx^2 + fx^3 = 0$.

Then by the Rational Theorem,

$$x \left(= \frac{b}{c + \frac{bd}{c}}\right) = -.02910318169;$$

Therefore $g + x = 357 - .02910318169 = 356.9708968183 = a$ very near.

And by the Irrational Theorem,

$$x \left(= \frac{\overline{bd + \frac{cc}{4}}\,\Big|^{\frac{1}{2}} - \frac{c}{2}}{d}\right) = -.029103181 80;$$

Therefore $g + x = 356.9708968182 = a$ very near.

Example III.

Let it be required to find one of the Affirmative Values of a in this Adfected Biquadratick Æquation; *Viz.*

$$a^4 - 80a^3 + 1998a^2 - 14937a + 5000 = 0.$$

That this Æquation may be the eaſier manag'd, ſuppoſe $\mathrm{a} = \frac{1}{10}a$, then the foregoing Æquation will become near

$$= \mathrm{a}^4 - 8\mathrm{a}^3 + 20\mathrm{a}^2 - 15\mathrm{a} + .5 = 0.$$

And for the firſt Suppoſition, let $g = 1$;

$$\left.\begin{array}{rl} \text{Then } \overline{g+x}|^4 = & 1+4x+6x^2+4x^3+x^4 \\ -8\times\overline{g+x}|^3 = & -8-24x-24x^2-8x^3 \\ +20\times\overline{g+x}|^2 = & 20+40x+20x^2 \\ -15\times\overline{g+x} = & -15-15x \\ +N = & +.5 \end{array}\right\}=0$$

$$-1.5+5x+2x^2-4x^3+x^4=0$$

That is $-b+cx+dx^2+fx^3+hx^4=0$.

Then by Dr. *Halley*'s Rational Theorem,

$$x\left(=\frac{b}{c+\frac{bd}{c}}\right)=\frac{1.5}{5+\frac{3}{5}}=.26\text{; wherefore } g+x=$$

1.26.

Or by his Irrational Theorem,

$$x\left(=\frac{\overline{bd+\frac{1}{4}cc}|^{\frac{1}{2}}-\frac{1}{2}c}{d}\right)=\frac{\overline{3+6\frac{1}{4}}|^{\frac{1}{2}}-2\frac{1}{2}}{2}=0.27.$$

Whence 'tis manifeſt, that 1.26, or 1.27 is near $= a$, and conſequently 12.6, or 12.7 near $= a$.

Now let us ſuppoſe $g = 12.7$;

$$\left.\begin{array}{rl} \text{Then } \overline{g+x}|^4 = & 26014.4641+8193.532x+ \\ & 967.74x^2+50.8x^3+x^4 \\ -80\times\overline{g+x}|^3 = & -163870.64-38709.6x- \\ & 3048x^2-80x^3 \\ +1998\times\overline{g+x}|^2 = & 322257.42+50749.2x+ \\ & 1998x^2 \\ -14937\times\overline{g+x} = & -189699.9-14937x \\ +N = & 5000 \end{array}\right\}=0$$

$$\begin{array}{l} -298.6559+5296.132x- \\ -b+cx+ \\ 82.26x^2-29.2x^3+x^4=0 \\ dxx+fx^3+hx^4=0 \end{array}$$

Then

Then by the Rational Theorem,

$$x\left(=\frac{b}{c+\frac{bd}{c}}\right)=\frac{298.6559}{5296.132-\frac{24567.434334}{5296.132}}=$$

$.05644$; therefore $g+x=12.75644$, which is near $=a$;

Or by Dr. *Halley*'s Irrational Theorem,

$$x\left(=\frac{\overline{bd+\frac{1}{4}cc}|^{\frac{1}{2}}-\frac{1}{2}c}{d}\right)=\frac{\sqrt{6987686.106022}-2648.066}{-82.26}$$

$=.05644080331$, which really is leſs than the true Value of x; but in order to Correct it, Subtract $\frac{\frac{1}{2}fx^3+\frac{1}{2}bx^4}{\sqrt{bd+\frac{1}{4}cc}}$ $=-\frac{.0026201}{2643.423}=-.00000099117$ from it, and you will have $.0564417 9448=x$ Corrected.

And if you deſire yet more Figures of the Root; from the x Corrected, let there be made $-d\times\overline{fx^3+hx^4}=-.43105602423$, and $\frac{-\frac{1}{2}c+\sqrt{bd+\frac{1}{4}cc-dfx^3-dhx^4}}{d}$

$$=\frac{-2648.066+\sqrt{6987685.67496597577}}{-82.26}=$$

$.056441794480744 02=x$.

Wherefore $g+x=12.7564417944807440 2=a$ very near.

The Reaſon or Demonſtration of the foregoing Corrections is this; *viz.* By what has been already ſaid and done, any Æquation may be reduc'd $=$ the following one, *viz.*

$$-b+cx+dx^2+fx^3+hx^4+\&c.=0.$$

And by Tranſpoſition and Diviſion,

$$xx+\frac{c}{d}x=\frac{b-fx^3-bx^4-\&c.}{d}$$

By Comp. □,

$$x^2 + \frac{c}{d}x + \frac{cc}{4dd} = \frac{cc}{4dd} + \frac{b - fx^3 - bx^4 - \&c.}{d} = \frac{\frac{1}{4}cc + db - dfx^3 - dbx^4 - \&c.}{dd}$$

By Evolution,

$$x + \frac{c}{2d} = \frac{\sqrt{\frac{1}{4}c^2 + db - dfx^3 - dbx^4 - \&c.}}{d}$$

And by Tranſpoſition,

$$x = \frac{-\frac{1}{2}c + \sqrt{\frac{1}{4}cc + db - dfx^3 - dbx^4 - \&c.}}{d}$$

Which is the laſt Correction.

And by Sir *Iſaac Newton*'s Theorem,

$$\sqrt{\frac{1}{4}cc + db - dfx^3 - dbx^4 - \&c.} \text{ is } = \sqrt{\frac{1}{4}cc + db} - \frac{\frac{1}{2}dfx^3 + \frac{1}{2}dbx^4 + \&c.}{\sqrt{\frac{1}{4}cc + db}}, \ \&c.$$

Therefore $\dfrac{-\frac{1}{2}c + \sqrt{\frac{1}{4}cc + bd - dfx^3 - dbx^4 - \&c.}}{d}$

$$= \frac{-\frac{1}{2}c + \sqrt{\frac{1}{4}cc + db}}{d} - \frac{\frac{1}{2}fx^3 + \frac{1}{2}bx^4 + \&c.}{\sqrt{\frac{1}{4}cc + bd}}, \ \&c.$$

Now the Value of x being very ſmall, this laſt &c. may be rejected, as being abundantly leſs than $\dfrac{\frac{1}{2}fx^3 + \frac{1}{2}bx^4 + \&c.}{\overline{\frac{1}{4}cc + bd}|^{\frac{1}{2}}}$, and ſo you have the firſt Correction.

CHAP.

CHAP. III.

Of *LOGARITHMS.*

DEFINITION.

See them otherwiſe defin'd in the latter Part of Ab. de Moivre's *Theorem.*

Logarithms are a Sett or Rank of artificial Numbers, accommodated to a Sett of natural Numbers, in ſuch ſort, that the Sum of the Logarithms (or artificial Numbers) of any two (natural) Numbers is equal to the Logarithm of of the Product of the ſaid two (natural) Numbers: And conſequently the Remainder of the Logarithms of any two Numbers, is equal to the Logarithm of the Quotient of the ſaid two Numbers; as alſo twice three times, four times, &c. or one half, one third, one fourth, &c. the Logarithm of any Number is equal to the Logarithm of the Square, Cube, Biquadrat, &c. or of the Square-Root, Cube-Root, Biquadrat-Root, &c. of the ſaid Number reſpectively.

LEMMA.

If you extract any Indefinit-Root out of each of two given Numbers, and from each of thoſe two Roots you ſubtract 1; I ſay the Sum of the two Indefinitely little Remainders is equal to the Indefinitely little Remainder of the ſaid Indefinite Root of the Product of the ſaid two given Numbers, an Unit being from it ſubducted.

In Order to Demonſtrate this *Lemma*, I will ſuppoſe any Indefinite Number to be $= n$.

DEMONSTRATION.

Either of the two given Numbers muſt be 1 or not.

1. If either of the given Numbers be 1, and the other $=$ (ſuppoſe) a, then,

$\overline{1}|^{\frac{1}{n}} - 1 \; (= 0) + \overline{a}|^{\frac{1}{n}} - 1 = \overline{1 \times a}|^{\frac{1}{n}} - 1$, which is the firſt thing I propos'd to prove.

2. If neither of the two given Numbers be 1, it will be requiſit to ſuppoſe one of them = ſome Binomial or Reſidual, whoſe firſt Member is 1; as (ſuppoſe) $1 \pm x$, and the other Number = ſome (finite) Power of $1 \pm x$, as (ſuppoſe) $\overline{1 \pm x}|^{m}$: then, altho' the Value of x be determin'd, the Value of $\overline{1 \pm x}|^{m}$ will be equal to any Number you pleaſe, but 1, ſince x is not = 0, and m is indetermin'd.

Thus $\overline{1 \pm x}|^{m}$ is $= a$, if m be $= \dfrac{\text{Logarithm of } a}{\text{Logarithm of } 1 \pm x}$.*

Now what we are to Demonſtrate in this 2d Caſe, is that

$$\overline{1 \pm x}|^{\frac{1}{n}} - 1 + \overline{1 \pm x}|^{\frac{m}{n}}\,{}^{*} - 1 \text{ is } = \overline{\overline{1 \pm x} \times \overline{1 \pm x}|^{m}}|^{\frac{1}{n}} - 1 = \overline{1 \pm x}|^{\frac{m+1}{n}} - 1.$$

*For $\overline{1 \pm x}|^{m}|^{\frac{1}{n}}$ is $= \overline{1 \pm x}|^{\frac{m}{n}}$.

By Sir *Iſaac Newton*'s Theorem,

$$1 \left\{ \overline{1 \pm x}|^{\frac{m}{n}} \text{ is } = 1 \pm \frac{m}{n}x + \frac{m^2 - mn}{2n^2}x^2 \pm \frac{m^3 - 3m^2n + 2mn^2}{6n^3}x^3 + \frac{m^4 - 6m^3n + 11m^2n^2 - 6mn^3}{24n^4}x^4 \right.$$

&c.

But ſince n is (by Suppoſition) Indefinite, any finite Number being added to, or taken from it, will not ſenſibly increaſe or diminiſh it: In like manner any finite Number of times n being added to, or ſubtracted from n^2, or any finite Number of times n^2 being added to, or taken from n^3, &c. will not ſenſibly, that is, will only by way of ϕ increaſe or diminiſh it: Wherefore,

* For if m be $= \dfrac{\text{Log. of } a}{\text{Log. of } 1 \pm x}$, then is $m \times$ Log. of 1±

= Log. of a: But (by latter Part of preceed. Defin.)

$m \times$ Log. of $1 \pm x$ is = Log. of $\overline{1 \pm x}|^{m}$;

and ſo Log. of $\overline{1 \pm x}|^{m}$ = Log. of a:

$\overline{1 \pm x}|^{\frac{m}{n}}$

Therefore is $\overline{1 \pm x}|^{m} = a$.

2 $\overline{1 \pm x}|^{\frac{m}{n}} \left(= 1 \pm \frac{m}{n}x + \frac{m^2 - mn}{2n^2}x^2 \pm \frac{m^3 - 3m^2n + 2mn^2}{6n^3}x^3 + \frac{m^4 - 6m^3n + 11m^2n^2 - 6mn^3}{24n^4}x^4, \text{\&c.}\right)$ is $= 1 \pm \frac{m}{n}x - \frac{mn}{2n^2}x^2 \phi \pm \frac{2mn^2}{6n^3}x^3 \phi - \frac{6mn^3}{24n^4}x^4 \phi$, &c. ϕ $= 1 \pm \frac{m}{n}x - \frac{m}{2n}x^2 \pm \frac{m}{3n}x^3 - \frac{m}{4n}x^4$, &c. ϕ

Therefore,

3 $\overline{1 \pm x}|^{\frac{m}{n}} - 1 = \pm \frac{m}{n}x - \frac{m}{2n}x^2 \pm \frac{m}{3n}x^3 - \frac{m}{4n}x^4 \pm \frac{m}{5n}x^5$, &c. ϕ.

From the 3d Step it is manifeſt that

4 $\overline{1 \pm x}|^{\frac{1}{n}} - 1$ is $= \pm \frac{1}{n}x - \frac{1}{2n}x^2 \pm \frac{1}{3n}x^3 - \frac{1}{4n}x^4$ &c. ϕ. by substituting 1 for m.

5 Alſo that $\overline{1 \pm x}|^{\frac{m+1}{n}} - 1$ is $= \pm \frac{m+1}{n}x - \frac{m+1}{2n}x^2 \pm \frac{m+1}{3n}x^3 - \frac{m+1}{4n}x^4$, &c. ϕ, which is equal to the Sum of the 3d and 4th Steps, that is,

$\overline{1 \pm x}|^{\frac{m+1}{n}} * - 1 = \overline{1 \pm x}|^{\frac{m}{n}} - 1 + \overline{1 \pm x}|^{\frac{1}{n}} - 1$. *Q.E.D.*

SCHOLIUM.

Hence if 1 be ſubtracted from the Indefinite Root of any Number, the Remainder may be taken for the Logarithm of the ſaid Number, had it not been indefinitely little; but if it be multiplied by 10000 &c. indefinitely, (which 10000 &c. indefinitely in the following Operations, is ſuppos'd to conſiſt of as many Figures as n doth) the Product will be Finite, and as the ſaid Remainder; that is to ſay, the Product will be the Logarithm required: As for Example.

* For $\overline{1 \pm x}|^{\frac{m+1}{n}}$ is $= \overline{1 \pm x}|^{\frac{m}{n}} \times \overline{1 \pm x}|^{\frac{1}{n}}$.

RULE.

RULE. I.

If the Logarithm of any Number $= b$, that is, if the Logarithm of the Ratio of 1 to any Number $= b$, be requir'd.

1. For the Value of b ſubſtitute $1 \pm x$; then, by what has been already ſaid, the Log. of $1 \pm x$ (or of b) is $= \overline{10000 \text{ \&c. indefinitely}} \times \overline{\pm \frac{1}{n}x - \frac{1}{2n}x^2 \pm \frac{1}{3n}x^3 - \frac{1}{4n}x^4, \text{ \&c.}}$

$$= \frac{10000 \text{ \&c. indefinitely}}{n} \times \overline{\pm x - \tfrac{1}{2}x^2 \pm \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4, \text{\&c.}}$$

$= \beta \times \overline{\pm x - \frac{1}{2}x^2 \pm \frac{1}{3}x^3 - \frac{1}{4}x^4 \pm \frac{1}{5}x^5, \text{\&c.}}$ putting $\beta = \frac{10000 \text{ \&c. indefinitely}}{n}$.

By this you may ſee that Logarithms may be of as many different Forms as the indefinite Number $= n$ can; Thus if n be ſuppos'd $= 10000$ &c. indefinitely, then β will $= 1$, and the Logarithm of $1 \pm x = \pm x - \frac{1}{2}x^2 \pm \frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. which is the Lord *Napier*'s Logarithm of $1 \pm x$; But if n be ſuppos'd $= 2302585$ &c. indefinitely, then β will be $= .434294$ &c. and the Logarithm of $1 \pm x = \overline{.434294 \text{ \&c.}} \times \overline{\pm x - \frac{1}{2}x^2 \pm \frac{1}{3}x^3 - \frac{1}{4}x^4}$, &c. which is Mr. *Brigs*'s Logarithm of $1 \pm x$.

Again, Putting the Terms of any Ratio $= r$ and s, of which s is the greater; and $s + r = z$, and $s - r = d$, we have the Logarithm of $\frac{s}{\frac{1}{2}z}$ (= Logarithm of $\frac{s}{\frac{1}{2}s + \frac{1}{2}r} =$ * Logarithm of $1 + \frac{\frac{1}{2}s - \frac{1}{2}r}{\frac{1}{2}s + \frac{1}{2}r} =$ Logarithm of $1 + \frac{\frac{1}{2}d}{\frac{1}{2}z}$) $=$ Logarithm of $1 + \frac{d}{z}$, which is (by Rule the firſt) $= \beta \times \overline{\frac{d}{z} - \frac{dd}{2zz} + \frac{d^3}{3z^3} - \frac{d^4}{4z^4}, \text{\&c.}}$

$$* \frac{\frac{1}{2}s+\frac{1}{2}r+\frac{1}{2}s-\frac{1}{2}r}{\frac{1}{2}s+\frac{1}{2}r} = \frac{\frac{1}{2}s+\frac{1}{2}r}{\frac{1}{2}s+\frac{1}{2}r} + \frac{\frac{1}{2}s-\frac{1}{2}r}{\frac{1}{2}s+\frac{1}{2}r} \overset{\text{And}}{=}$$

And the Logarithm of $\frac{r}{\frac{1}{2}z}$ (= Logarithm of $\frac{r}{\frac{1}{2}r+\frac{1}{2}s}$ = Logarithm of $1+\frac{\frac{1}{2}r-\frac{1}{2}s}{\frac{1}{2}r+\frac{1}{2}s}$ = Logarithm of $1-\frac{\frac{1}{2}d}{\frac{1}{2}z}$) = Logarithm of $1-\frac{d}{z}=\beta\times\overline{-\frac{d}{z}+\frac{dd}{2zz}+\frac{d^3}{3z^3}+\frac{d^4}{4z^4}+\&c.}$

Now if from the Logarithm of $\frac{s}{\frac{1}{2}z}$ you ſubtract the Logarithm of $\frac{r}{\frac{1}{2}z}$ you will have

RULE II.

The Logarithm of $\frac{r}{\frac{1}{2}z}$) $\frac{s}{\frac{1}{2}z}$ (= Logarithm of $\frac{s}{r}=\beta\times\overline{\frac{2d}{z}*+\frac{2d^3}{3z^3}*+\frac{2d^5}{5z^5}*+\&c.}=2\beta\times\overline{\frac{s-r}{s+r}+\frac{1}{3}\times\left.\overline{\frac{s-r}{s+r}}\right|^3+\frac{1}{5}\times\left.\overline{\frac{s-r}{s+r}}\right|^5+\&c.}=A+\frac{1}{3}B+\frac{1}{5}C+\frac{1}{7}D+\&c.$ putting $2\beta\times\overline{\frac{s-r}{s+r}}=A$, $2\beta\times\left.\overline{\frac{s-r}{s+r}}\right|^3=B$, $2\beta\times\left.\overline{\frac{s-r}{s+r}}\right|^5=C$. &c. which is an excellent Rule for finding the Logarithms of ſmall Numbers: But in order to find the Logarithm of a great Number by it, it will be very convenient to conſider the ſaid great Number, as the Product of ſome ſmall Numbers, ſuch, whoſe Logarithms are moſt of them (or all, if you can) already known: For the nearer any Number is to Unity, the ſwifter its Logarithm (when to be found by either of the two foregoing Rules) will Converge.

Again, by adding the Logarithms found by Rule 1ſt of $\frac{s}{\frac{1}{2}z}$ and $\frac{r}{\frac{1}{2}z}$ we have the Log. of $\frac{s}{\frac{1}{2}z} \times \frac{r}{\frac{1}{2}z} = \beta \times -\overline{\frac{2dd}{2zz} + \frac{2d^4}{4z^4} + \frac{2d^6}{6z^6}}$, &c. and by ſubtracting this Logarithm, from the Log. of 1, that is from 0, we have the Log. of $\frac{sr}{\frac{1}{4}zz}$) 1 (= Log. of $\frac{\frac{1}{4}zz}{sr} = \beta \times + \overline{\frac{2dd}{2zz} + \frac{2d^4}{4z^4} + \frac{2d^6}{6z^6} + \&c}$ *
And half this Logarithm, *viz.*

RULE. III.

$\beta \times \overline{\frac{dd}{2zz} + \frac{d^4}{4z^4} + \frac{d^6}{6z^6} + \frac{d^8}{8z^8} + \&c.}$ is = Log. of $\frac{\frac{1}{2}z}{\sqrt{rs}}$ *i. e.* of the Ratio of the Geometrical, to the Arithmetical Mean between r and s.

And $\frac{\frac{1}{4}zz}{rs}$ ($= \frac{\frac{1}{4}rr + \frac{1}{2}rs + \frac{1}{4}ss}{rs} = \frac{\frac{1}{4}r^2 - \frac{1}{2}rs + \frac{1}{4}ss + rs}{rs}$) is $= \frac{\frac{1}{4}dd + rs}{rs}$; and its Logarithm is found (by ſubſtituting $\frac{1}{4}dd + rs$ inſtead of s, and rs inſtead of r in *Rule* 2.) $= 2\beta \times \overline{\frac{\frac{1}{4}dd + rs - rs}{\frac{1}{4}dd + rs + rs} + \frac{1}{3} \times \overline{\frac{\frac{1}{4}dd}{\frac{1}{4}dd + 2rs}}\Big|^3 + \frac{1}{5} \times \overline{\frac{\frac{1}{4}dd}{\frac{1}{4}dd + 2rs}}\Big|^5}$ + &c. that is (putting $1 = \frac{1}{4}dd$, and $y = \frac{1}{4}dd + 2rs$) $= 2\beta \times \overline{\frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7}} + \&c.$

* For whatever is the Logarithm of a first Number divided by a ſecond, is alſo (with a contrary Sign) the

Wherefore the Logarithm of $\frac{\frac{1}{2}z}{\sqrt{rs}}$ (*i. e.* of the Ratio of $\sqrt{rs}$ to $\frac{1}{2}z$) is $= \beta \times \overline{\frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} + \&c.}$ $= \text{Log. of } \frac{1}{2}z - \text{Log. of } \sqrt{rs} = \text{Log. of } \frac{1}{2}z - \frac{L,r + L,s}{2}$ (by the nature of Logarithms): Conſequently,

RULE IV.

The Log. of $\frac{1}{2}z$ is $= \frac{L,r + L,s}{2} + \overline{\beta \times \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} + \&c.} = \frac{L,r + L,s}{2} + A + B + C + D + \&c.$ putting $\beta \times \frac{1}{y} = A$, $\frac{1}{3yy} A = B$, $\frac{3}{5yy} B = C$, $\frac{5}{7yy} C = D$, &c. which (*Rule* 4.) is of excellent Uſe for finding the Logarithms of prime Numbers, having the Logarithms of the adjoyning Numbers given.

Note that $\frac{1}{2}z$ *is always* = *Number*, *whoſe Logarithm you ſeek by* Rule 4. *and* (*ſince* $\frac{1}{4}dd = 1$; *therefore* $d = 2$ *and*) *conſequently* $\frac{1}{2}z + 1 = s$, *and* $\frac{1}{2}z - 1 = r$.

Examples.

1. Let it be required to find *Napier*'s Logarithm of 2 to 13 Places of Figures.

Suppoſe $r = 1$, and then s will be $= 2$, and (by *Rule* 2.) *Napier*'s Logarithm of 2 will be (ſince β, in this Caſe, is $= 1$) $= 2 \times \overline{\frac{2-1}{2+1} + \frac{1}{3 \times 3^3} + \frac{1}{5 \times 3^5} + \&c.}$ Wherefore the Operation will very advantagiouſly ſtand thus;

$*2 \times \frac{1}{3} = A = .66666666666666$
$\frac{1}{9}A = B = 7407407407407$
$\frac{1}{9}B = C = 823045267489$
$\frac{1}{9}C = D = 91449474165$
$\frac{1}{9}D = E = 10161052685$
$\frac{1}{9}E = F = 1129005853$
$\frac{1}{9}F = G = 125445094$
$\frac{1}{9}G = H = 13938343$
$\frac{1}{9}H = I = 1548704$
$\frac{1}{9}I = K = 172078$
$\frac{1}{9}K = L = 19119$
$\frac{1}{9}L = M = 2124$
$\frac{1}{9}M = N = 236$

Therefore,

$A = .66666666666666$
$\frac{1}{3}B = 2469135802469$
$\frac{1}{5}C = 164609053497$
$\frac{1}{7}D = 13064210595$
$\frac{1}{9}E = 1129005853$
$\frac{1}{11}F = 102636895$
$\frac{1}{13}G = 9649622$
$\frac{1}{15}H = 929222$
$\frac{1}{17}I = 91100$
$\frac{1}{19}K = 9056$
$\frac{1}{21}L = 910$
$\frac{1}{23}M = 92$
$\frac{1}{25}N = 9$

Napier's Log. of 2 = .69314718055994

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* *See* Rule 2.

Example

Example. II.

Let it be required to find the Value of the Index *n*, in Order to make *Briggs*'s Logarithms, the ſaid *Briggs*'s Logarithm of 10 being 1.

The Queſtion propos'd is the ſame with this; *viz.*

Briggs's Log. of 10 *

$$\frac{10000\ \&c.\ \text{Indefinitely}}{n} \times \frac{2}{1} \times \overline{\frac{10-1}{10+1} + \frac{2}{3} \times \left.\overline{\frac{9}{11}}\right|^{3} + \frac{2}{5} \times \left.\overline{\frac{9}{11}}\right|^{5} + \&c.} = 1$$

that is (by multiplying each Part by $\frac{n}{10000\ \&c.\ \text{Indefinitely}}$) $2 \times \overline{\frac{9}{11} + \frac{1}{3} \times \left.\overline{\frac{9}{11}}\right|^{3} + \frac{1}{5} \times \left.\overline{\frac{9}{11}}\right|^{5} + \&c.}$ $= \frac{n}{10000\ \&c.\ \text{Indefinitely}}$; that is in Words, *Napiers* Logarithm of 10 being $= \frac{n}{10000\ \&c.\ \text{Indefinitely}}$, what is *n* equal to?

Now becauſe $2 \times \overline{\frac{9}{11} + \frac{1}{3} \times \left.\overline{\frac{9}{11}}\right|^{3} + \frac{1}{5} \times \left.\overline{\frac{9}{11}}\right|^{5} + \&c.}$ converges very ſlow, it will be convenient to conſider 10 as compos'd of the Product of ſome Power of a leſs Number, multiplied by ſome other leſs Number; ſo 10 may be very advantagiouſly conſider'd as $= 2^{3} \times 1\frac{1}{4}$: Then $3 \times$ *Napier*'s Log. of 2 + *Napier*'s Log. of $1\frac{1}{4} = \frac{n}{10000\ \&c.\ \text{Indefinitely}}$: But $3 \times$ *Napier*'s Logarithm of 2 is (by *Example* 1.) = 2.0794415416797; and *Napier*'s Log. of $1\frac{1}{4}$ may be found thus:

* this follows from 2d Rule, because there β is $= \frac{1000\ \&c.\ \text{indefinitely}}{n}$; and $s = 10$, $r = 1$.

$1\frac{1}{4}$

$\frac{1\frac{1}{4}-1}{1\frac{1}{4}+1}=\frac{1}{9}$; wherefore *Napier's* Log. of $1\frac{1}{4}$ is (by *Rule* 2.)

$$=2\times\overline{\frac{1}{9}+\frac{1}{3\times9^3}+\frac{1}{5\times9^5}}+\&c.$$

OPERATION.

$$\begin{array}{rcl}
2\times\frac{1}{9} & =A= & .22222222222222 \\
\frac{1}{81}A & =B= & 274348422496 \\
\frac{1}{81}B & =C= & 3387017561 \\
\frac{1}{81}C & =D= & 41815031 \\
\frac{1}{81}D & =E= & 516234 \\
\frac{1}{81}E & =F= & 6373 \\
\frac{1}{81}F & =G= & 78
\end{array}$$

Therefore,

$$\begin{array}{rcr}
A & = & .22222222222222 \\
\frac{1}{3}B & = & 91449474165 \\
\frac{1}{5}C & = & 677403512 \\
\frac{1}{7}D & = & 5973575 \\
\frac{1}{9}E & = & 57359 \\
\frac{1}{11}F & = & 579 \\
\frac{1}{13}G & = & 6
\end{array}$$

Napier's Log. of $1\frac{1}{4}$ = .22314355131421+
3 × *Napier's* Log. of 2 = 2.0794415416797

Napier's Log. of 10 = 2.302585092994+

$=\frac{n}{10000\ \&c.\ \text{Indefinitely}}$: and by multiplying each part by 10000 &c. Indefinitely, we have 2302585092994 &c. Indefinitely = n.

Example III.

Let it be required to find *Briggs*'s Logarithm of 2, to 8 Places.

Note, That the Index $= \beta$ muſt have a Figure or two, at leaſt, more than the intended Logarithm is to have; therefore in this *Example* 3. it muſt have 9 or 10 Figures in it; that is β $\left(= \frac{10000 \text{ \&c.}}{23025 \text{ \&c.}}\right) = .4342944819$.

And (ſuppoſing $r = 1$) $\overline{2\beta \times \frac{1}{3} + \frac{1}{3} \times \overline{\frac{1}{3}}\Big|^{3} + \frac{1}{5} \times \overline{\frac{1}{3}}\Big|^{5} + \text{\&c.}}$ is (by *Rule* 2.) $=$ *Briggs*'s Logarithm of 2.

OPERATION.

$2\beta \times \frac{1}{3} = A =$	.289529654	$A =$	.289529654
$\frac{1}{9}A = B =$	32169961	$\frac{1}{3}B =$	10723320
$\frac{1}{9}B = C =$	3574440	$\frac{1}{5}C =$	714888
$\frac{1}{9}C = D =$	397160	$\frac{1}{7}D =$	56737
$\frac{1}{9}D = E =$	44128	$\frac{1}{9}E =$	4903
$\frac{1}{9}E = F =$	4903	$\frac{1}{11}F =$	445
$\frac{1}{9}F = G =$	544	$\frac{1}{13}G =$	41
$\frac{1}{9}G = H =$	60	$\frac{1}{15}H =$	4

Briggs's Logarithm of 2 $= .30102999$

Example IV.

Let it be required to find *Briggs*'s Logarithm of 3, to 8 Places.

By *Rule* 2. *Briggs*'s Log. of 3 is $= 2\beta \times \overline{\frac{3-1}{3+1} + \frac{1}{3} \times \frac{2}{4}\Big|^{3} + \frac{1}{5} \times \frac{1}{2}\Big|^{5}} + \text{\&c.}$ But ſince *Briggs*'s Logarithm of 2 is already

ready known, 'tis eaſier to find *Brigg*'s Logarithm of 3, by conſidering 3 as $= 2 \times 1\frac{1}{2}$, and then finding *Brigg*'s Log. of $1\frac{1}{2}$, which is by *Rule* 2.) $=$

$$2\beta \times \overline{\frac{1\frac{1}{2}-1}{1\frac{1}{2}+1} + \frac{1}{3} \times \frac{1}{5^3} + \frac{1}{5} \times \frac{1}{5^5}} \text{ \&c.}$$

OPERATION.

$$\begin{array}{rcl|rl}
2\beta \times \frac{1}{5} = A = & .173717792 & & A = & .173717792 \\
\frac{1}{25} A = B = & 6948711 & & \frac{1}{3} B = & 2316237 \\
\frac{1}{25} B = C = & 277948 & & \frac{1}{5} C = & 55589 \\
\frac{1}{25} C = D = & 11117 & & \frac{1}{7} D = & 1588 \\
\frac{1}{25} D = E = & 444 & & \frac{1}{9} E = & 49 \\
\frac{1}{25} E = F = & 17 & & \frac{1}{11} F = & 1
\end{array}$$

Briggs's Logarithm of $1\frac{1}{2} = .17609125$

Brigg's Logarithm of $2 = .30102999$

Brigg's Logarithm of $3 = .47712125$

Or ſince *Brigg*'s Logarithm of 2, and therefore of 4 (2^2) is already known, *Brigg*'s Logarithm of 3 may be ſooner found by *Rule* 4. thus,

$\frac{1}{2}z$ being $= 3$, and $d = 2$; then s will be $= 4$, and $r = 2$; and therefore $\frac{1}{4}zz + rs = 9 + 8 = 17 = y$; conſequently (by *Rule* 4.) *Brigg*'s Logarithm of 3 is $=$

$$\frac{L,2 + L,4}{2} + \beta \times \overline{\frac{1}{17} + \frac{1}{3 \times \overline{17}|^3} + \frac{1}{5 \times \overline{17}|^5} + \text{\&c.}}$$

OPERATION.

$$\frac{L,2+L,4}{2} = .451544993$$

$$* \beta \times \frac{1}{17} = A = 25546734$$

$$\frac{1}{3 \times 289} A = B = 29465$$

$$\frac{3}{5 \times 289} B = C = 61$$

Brigg's Logarithm of 3 = .47712125$\underline{|\cdot}$

Example V.

Let it be required to find *Brigg*'s Logarithm of the prime Number 17, by *Rule* 4.

Here $\frac{1}{2} z$ being $= 17$, and $d = 2$; r will be $= 16$, and $s = 18$; and therefore $\frac{1}{4} zz + rs = 577 = y$;

And the Logarithm of $16 = 4 \times L, 2$, and the Log. of $18 = 2 + 2 * L, 3$ are given by *Examples* 3 and 4; wherefore, by *Rule* 4.

OPERATION.

$$\frac{L,16+L,18}{2} = 1.22969624$$

$$\beta \times \frac{1}{577} = A = 75267$$

Brigg's Log. of 17 = 1.2304489$\underline{|\cdot}$

SCHOLIUM.

Hence also the Logarithm of any Number being given along with the Index, *n*, the Number it ſelf may be found thus,

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* *See* Rule 4.

$\overline{1 \pm x}|^{\frac{1}{n}} - 1 \times 10000$ &c. Indefinitely is = Logarithm of $1 \pm x$; wherefore $\overline{1 \pm x}|^{\frac{1}{n}} - 1 = \frac{\text{Log. of } 1 \pm x}{10000 \text{ \&c. Indefinitely}}$ which ſuppoſe $= l$; then $\overline{1 \pm x}|^{\frac{1}{n}} = 1 + l$: and (by equal Involution) $1 \pm x = \overline{1 + l}|^{n} = 1 + nl + n \times \frac{n-1}{2} l^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} l^3$ &c. $= 1 + nl + \frac{n^2}{2} l^2 + \frac{n^3}{6} l^3 + \frac{n^4}{24} l^4$, &c. (becauſe n is Indefinite.)

Note, That as n is indefinitely great, ſo l is indefinitely little; and therefore $n\,l$, $\frac{n^2}{2} ll$, $\frac{n^3}{6} l^3$, &c. are = to finite Numbers.

But this Series converges very ſlow, eſpecially in great Numbers; wherefore when the Logarithm of any Number is given along with the Value of the Index n, and the Number it ſelf required, your better way is this, *viz.* Seek in the Table of Logarithms the * next neareſt Logarithm, without their Characteriſticks, to that which is given, and call its Number r if leſs, or s if greater than the given Logarithm, and the Number ſought, or that whoſe Logarithm is given, ſuppoſe $= u$: And if the Logarithm of s or r in the ſaid Table be not exact enough for your Purpoſe, you may, by ſome of the four foregoing Rules find it to what exactneſs you pleaſe.

Then the Logarithm of u − Logarithm of r, or Logarithm of s − Logarithm of u = Logarithm of $\frac{u}{r}$ or of $\frac{s}{u}$ is (by what has been already ſaid in this *Chap.*) = $\overline{\frac{u}{r}}|^{\frac{1}{n}} - 1 \times 10000$ &c. Indefinitely or $= \overline{\frac{s}{u}}|^{\frac{1}{n}} - 1 \times 10000$

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* *Thus the next neareſt Logarithm to* .8526627 *in the Table of Logarithms for any Number not exceeding* 10000 *is* .8526629, *whoſe Natural Number is* $7,123 = \frac{7123}{1000}$.

&c. Indefinitely; wherefore $\frac{\text{the Log. of } \frac{u}{r}}{10000 \text{ \&c. Indefinitely}}$ or $\frac{\text{the Log. of } \frac{s}{u}}{10000 \text{ \&c. Indefinitely}}$ is $= \overline{\frac{u}{r}}\Big|^{\frac{1}{n}} - 1$, or $= \overline{\frac{s}{u}}\Big|^{\frac{1}{n}} - 1$ $= l$; wherefore $\overline{1+l}|^{n} = \frac{u}{r}$ or $= \frac{s}{u}$ is (by Sir *Iſaac Newton*'s Theorem, and ſince n is Indefinite) $= 1 + nl + \frac{n^2}{2} l^2 + \frac{n^3}{6} l^3 + \frac{n^4}{24} l^4$, &c. ϕ.

Whence u is $= r \times \overline{1 + nl + \frac{1}{2} n^2 l^2 + \frac{1}{6} n^3 l^3, \text{\&c.}\ \phi}$

Or u is $= \frac{s}{1 + nl + \frac{1}{2} n^2 l^2 + \frac{1}{6} n^3 l^3, \text{\&c.}\ \phi}$

Or ſince $\overline{1+l}|^{n}$ is $= \frac{s}{u}$;

Therefore $\overline{1+l}|^{n}\ )\ 1\ (= \overline{1+l}|^{-n}\ ) = \frac{s}{u}\ )\ 1\ (= \frac{u}{s}$

That is, by Sir *Iſaac Newton*'s Theorem, ſince n is Indefinite $1 - nl + n^2 l^2 - \frac{1}{6} n^3 l^3 + \frac{1}{24} n^4 l^4 - \frac{1}{120} n^5 l^5$, &c. ϕ $= \frac{u}{s}$.

Whence $u = s \times 1 - nl + \overline{nl}|^2 - \frac{1}{6} \times \overline{nl}|^3$, &c.

Again theſe Values of u may be contracted thus; for Inſtance, u being $= r + rnl + \frac{1}{2} rn^2 l^2 + \frac{1}{6} rn^3 l^3 + \frac{1}{24} rn^4 l^4 + \frac{1}{120} rn^5 l^5 + \frac{1}{720} rn^6 l^6 +$ &c. ϕ

Firſt find a Fractional Quantity, whoſe Numerator (in order to the deſign'd contraction) muſt conſiſt only of one Member as rnl, and the Denominator muſt have at leaſt two Members, but we will ſuppoſe the ſaid Denominator to have only two Members, which ſuppoſe $= a + e$. Now the two firſt Members of the Quotient of the ſaid Fractional Quantity

$\frac{rnl}{a+e}$, to wit $\frac{rnl}{a} - \frac{rnle}{aa}$ muſt be $= rnl + \frac{1}{2} rn^2 l^2$ reſpectively; whence a is $= 1$, and e is $= -\frac{1}{2} nl$; wherefore the required Fraction is $= \frac{rnl}{1 - \frac{1}{2} nl}$: And its Quotient is $= rnl + \frac{1}{2} rn^2 l^2 + \frac{1}{4} rn^3 l^3 + \frac{1}{8} rn^4 l^4 + \frac{1}{16} rn^5 l^5 + \frac{1}{32} rn^6 l^6 +$ &c. from which ſubtract the above Value of u, except its firſt Member r, and the Remainder is $\frac{1}{12} rn^3 l^3 + \frac{1}{12} rn^4 l^4 + \frac{13}{240} rn^5 l^5 + \frac{43}{440} rn^6 l^6 +$ &c. whence u is $= r + \frac{rnl}{1 - \frac{1}{2} nl}$; that is u is $= r + \frac{rnl}{1 - \frac{1}{2} nl} - \frac{1}{12} rn^3 l^3 - \frac{1}{12} rn^4 l^4 - \frac{13}{240} rn^5 l^5 - \frac{43}{440} rn^6 l^6 -$ &c.

Again the Denominator of another Fraction, whoſe Numerator is $\frac{1}{12} rn^3 l^3$ (the firſt Member of the foregoing Remainder) and the two firſt Members of whoſe Quotient are $\frac{1}{12} rn^3 l^3 + \frac{1}{12} rn^4 l^4$ will be found by the foregoing Method, to be $1 - nl$; and (therefore) the Quotient is $\frac{1}{12} rn^3 l^3 + \frac{1}{12} rn^4 l^4 + \frac{1}{12} rn^5 l^5 + \frac{1}{12} rn^6 l^6 +$ &c. from which ſubtracting the foregoing Remainder, there remains $\frac{7}{240} rn^5 l^5 + \frac{19}{1320} rn^6 l^6 +$ &c. wherefore the next Fraction ($\frac{7}{240} rn^5 l^5$ being its Numerator, and $\frac{7}{240} rn^5 l^5 + \frac{19}{1320} rn^6 l^6$ the two firſt Members of its Quotient) will be found (by proceeding as before taught) $= \frac{\frac{7}{240} rn^5 l^5}{1 - \frac{38}{77} nl}$ &c.

Whence u is $= r + \frac{rnl}{1 - \frac{1}{2}nl} - \frac{\frac{1}{12}rn^3l^3}{1 - nl} + \frac{\frac{7}{540}rn^5l^5}{1 - \frac{38}{77}nl}$, &c. Or u will be found (by a Proceſs like the foregoing) $= s - \frac{snl}{1 - \frac{1}{2}nl} + \frac{\frac{1}{12}sn^3l^3}{1 - nl} - \frac{\frac{7}{540}sn^5l^5}{1 - \frac{38}{77}nl}$, &c. the firſt Step of which, namely $u = r + \frac{rnl}{1 - \frac{1}{2}nl}$, or $u = s - \frac{snl}{1 - \frac{1}{2}nl}$ is an approach near enough for Practice.

> Note, *By how much the leſs the Logarithm of* u — *Logarithm of* r, *or the Logarithm of* s — *Logarithm of* u, *is, by ſo much the ſwifter the Series will converge.*

I will give you one Example of this *Scholium.*

* *Example.*

If $a^{365} = 1.06$; the Queſtion is what a is equal to?

The Value of a may be with a great deal of Facility, in reſpect of other Methods, found by Dr. *Halley*'s Rational Theorem for evolving the Roots of Æquations (of which I have treated in its proper Place, to wit in the Converging Series) by ſuppoſing $g = 1$; for by proceeding by the ſaid Theorem with the ſaid Value of g, the Value of x will be found $= .0001596$ and conſequently the Value of $g + x$ ($= g$ the 2d.) is $= 1.0001596$ which is pretty near equal to the Value of *a*; but if it be not near enough for your Purpoſe, renew the ſaid Theorem with the ſaid Value of *g* the 2d = 1.0001596, and the following *g* (or *g* the 3d) will have the true Figures in it to a great many Places, that is it will be near equal to the Value of *a*; but the Value of *a* may be more exactly had, or ſooner, by ſuppoſing $g = 1$, and by Dr. *Halley*'s Irrational Theorem and its Corrections, of which I have treated in the Converging Series.

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* *See its Uſe in the following* Part, Chap. 2.

But I will now ſhew how to find the Value of *a*, expeditiouſly and exactly enough, by the foregoing Scholium. Thus,

$a^{365} = 1.06$, by Hypotheſis; therefore (by the Nature of Logarithms) $365 \times \text{L}, a = \text{L}, 1.06$
$= .0253058652647702408467\,31$; wherefore

$$\text{L}, a = \frac{.025305865264770240846731}{365} = .000069331137\,711699289991.$$

Having thus found the Logarithm of *a*, *a* itſelf may be diſcover'd by the latter Part of our Scholium, by the help of the Table of Logarithms thus: Seek in the ſaid Table the neareſt Logarithm without its Characteriſtick to the above Logarithm of *a*, and you'll find it to be 4.0000434; whoſe abſolute Number in the ſaid Table is $1.0001 = r$: But the Logarithm of 1.0001 is = .0000434272768626669637313, which ſubtracted from the above Logarithm of *a*, the Remainder is .000025903860849029652678;

$$\text{Therefore } \frac{.000025903860849029652678}{10000 \text{ \&c. Indefinitely}} = l:$$

$$\text{And } u = a \text{ is } = r + \frac{rnl}{1 - \frac{1}{2}nl} - \frac{\frac{1}{12}rn^3l^3}{1 - nl}, \text{ \&c.}$$

$$= 1.0001 + \frac{.00000596518084263519587\,6919 \text{ \&c.}}{1 - .000002982292192098388\,09 \text{ \&c.}} -$$

0000

$$\frac{.0000000000000176849054637\ \&c.}{1 - .000059645843\ \&c.} = 1.0001 + .0000596535874706334020\ \&c. - .0000000000000176859603\ \&c. = 1.0001596535874529474417\ \&c.$$

N. B. *Part* 16. *begins* Page 289. Signature *P p.*

PART

PART XVI.

Of Intereſt.

CHAP. I.

Intereſt, or the Uſe paid for the Loan of Money, may be either Simple, or Compound.

Sect. 1. Of Simple Intereſt.

SImple Intereſt is that which is paid for the Loan of any Principal, or Sum of Money, lent out for ſome time at any Rate *per Cent.* agreed on between the Borrower and Lender; which, according to the Laws of *England*, ought to be but Six Pounds for the Uſe of 100 Pounds for one Year, and 12 Pounds for the Uſe of 100 Pounds for two Years, and ſo on for a greater or leſſer Sum, proportionable to the time propoſed.

Note, *That a great part of this Part* 16, *has been Tranſcribed out of Mr.* Ward's Young Mathematicians Guide, *to which this Book is indebted for ſome other things in it.*

There are ſeveral ways of computing (or anſwering Queſtions about) Simple Intereſt; as by the ſingle and double Rule of Three. Others make uſe of Tables compos'd at ſeveral Rates *per Cent.* As Sir *Samuel Moreland*, in his Doctrine of Intereſt, both Simple and Compound, is all perform'd by Tables, wherein he hath [*ſays my Author*] detected ſeveral material Errors committed by Dr. *Newton*, Mr. *Kerſey* upon *Wingat*, and Mr. *Clavill*, &c. in the Buſineſs of computing Intereſt, *&c.* by their Tables, too tedious to be here repeated.

But I ſhall in this Tract take other Methods, and ſhew, that all Computations relating to Simple Intereſt, are grounded upon Arithmetick Progreſſion; and from thence raiſe ſuch general Theorems as will ſuit with all Caſes. In order to that,

Let $\begin{cases} p = \text{any Principal or Sum put to Intereſt} \\ r = \text{the Ratio of the Rate per 1 } l. \text{ per Annum.} \\ t = \text{the time of the Principal's continuance at Intereſt.} \\ A = \text{the amount of the Principal, and its Intereſt.} \end{cases}$

Note, The Ratio of the Rate is only the ſimple Intereſt of 1 *l.* for one Year, at any given Rate, and is thus found,

Viz. 100 .. 6 :: 1 .. 0,06 = the Ratio at 6 *per Cent. per Ann.*

Or 100 .. 7 :: 1 .. 0, 07 = the Ratio at 7 *per Cent.* &c.

Again, 100 .. 7 . 5 :: 1 .. 0,75 = the Ratio at 7 $\frac{1}{2}$ *per Cent.* &c. And if the given time be whole Years; then t = the number of thoſe Years: But if the time given be either pure Parts of a Year, or parts of a Year mixed with Years; thoſe Parts muſt be turn'd to Decimals; and then t = thoſe Decimals, &c. Now the common parts of a Year may be eaſily turn'd, or converted into Decimal parts, if it be conſidered,

That one { Day is the $\frac{1}{365}$ part of a Year = 0,00274 *fere.*
Month is the $\frac{1}{12}$ part of a Year = 0,0833333, &c.
Quarter is the $\frac{1}{4}$ part of a Year = 0, 25.
Half Year = 0. 5: and three Quarters, = 0.75.

Theſe things being premiſed, we may proceed to raiſing the Theorems.

Let r = the Intereſt of 1 *l.* for one Year (as before)
then $2r$ = the Intereſt of 1 *l.* for two Years.
$3r$ = the Intereſt of 1 *l.* for three Years.
and $4r$ = the Intereſt of 1 *l.* for four Years:
And ſo on, for any number of Years propoſed.

Hence it is plain, that the Simple Intereſt of one Pound is a Series of Terms in Arithmetick Progreſſion Increaſing, whoſe firſt Term and common Difference is r, and the number of all the Terms is t; therefore the laſt Term will always be $= tr$ = the Intereſt of 1 *l.* for any given time ſignified by t.

Then { As one Pound .. is to the Intereſt of 1 *l.* :: ſo is any Principal or given Sum .. to its Intereſt;

That is, 1 *l.* . tr :: p .. ptr = the Intereſt of p: Then the Principal being added to its Intereſt, their Sum will be $= A$ the amount required: Which gives this general Theorem.

1. Theorem $trp + p = A = p \times \overline{tr+1}$.

From whence the three following Theorems are eaſily deduced.

Theorem 2. $\frac{A}{tr+1} = p$.

Theorem 3. $\frac{A-p}{tp} = r$.

Theorem 4. $\frac{A-p}{rp} = t$.

Theſe four Theorems reſolve all Queſtions about Simple Intereſt.

Queſt. 1. What will 256 *l.* 10 *s.* 00 *d.* amount to in three Years, one Quarter, two Months and eighteen Days, at 6 *per Cent. per Annum?*

Here is given $p = 256.5$, $r = 0.06$ and $t = 3.46599$;
For 3 Years $= 3.$ Quere *A.* *per Theorem* 1.
one Quarter $= 0.25$
2 Months $= 0.16667 = 0.08333 \times 2.$
18 Days $= \underline{0.04932} = 0.00274 \times 18.$

Hence $t = 3.46599$
And $tr = 3.46599 \times 0.06 = 0.2079594$
Then $0.2079594 \times 256.5 = 53.341586 = trp.$
And $53.341586 + 256.5 = 309.841586 = trp + p =$
$A =$ 309 *l.* 16 *s.* 10 *d.* *Anſ:*

Queſt. 2. What Principal or Sum being put to Intereſt will raiſe a Stock of 309 *l.* 16 *s.* 10 *d.* in 3 Years, 1 Quarter, 2 Months and 18 Days, at 6 *per Cent. per Annum?*

Or the ſame Queſtion otherwiſe Stated. Thus;

What is 309 *l.* 16 *s.* 10 *d.* due at 3 Years, 1 Quarter, 2 Months and 18 Days hence, worth in Ready Money, abating or diſcompting 6 *per Cent. per Annum* Simple Intereſt.

Here is given $A = 309.841586$
$r = 0.06$ (for $100 \cdot\cdot 6 :: 1 \cdot\cdot 0.06$)
$t = 3.46599$ (found as before) thence to find p *per* Theorem 2.

Firſt, $3.46599 \times 0.06 = 0.2079594 = tr.$
Then $tr + 1 = 1.2079594$
$1.2079594\,)\ 309.841586\ (\,256.5 = \frac{A}{tr+1} = p.$
$=$ 256 *l.* 10 *s.* 00 *d.* the Anſwer required.

Q. 3. At what Rate of Intereſt *per Cent. &c.* will 256 *l.* 10 *s.* amount to 309 *l.* 16 *s.* 10 *d.* in 3 Years, 1 Quarter, 2 Months and 18 Days?

Here is given $p = 256.5$, $A = 309.841586$, and $t = 3.46599$: To find r *per* Theorem 3.

Firſt, $309.841586 - 256.5 = 53.341586 = A - p$.

Next, $3.46599 \times 256.5 = 889.026435 = tp$.

And $889.026435 \,)\, 53.341586 \,(\, 0.06 = \frac{A-p}{tp} = r$.

Then 1 *l.* .. 0.06 :: 100 .. 6 the Rate required.

Q. 4. In what time will 256 *l.* 10 *s.* raiſe a Stock of (or amount to) 309 *l.* 16 *s.* 10 *d.* at 6 *per Cent.* &c.

Here is given $p = 256.5$; $A = 309.841586$

And $r = 0.06$ to find t *per* Theorem 4.

Firſt, $309.841586 - 256.5 = 53.341586 = A - p$.

And $256.5 \times 0.06 = 15.39 = pr$.

Then $15.39 \,)\, 53.341586 \,(\, 3.46599 = \frac{A-p}{pr} = t$;

That is $t = 3$ Years, and .46599 Decimal parts of a Year; which may be brought into the common parts of a Year; Thus,

1 Quarter = .25) .46599 (1 Quarter
1 Month = .0833) .21599 (2 Months
1 Day = .00274) .04933 (18 Days.

Hence $t = 3$ Years, 1 Quarter, two Months 18 Days, the Anſwer required.

It muſt needs be eaſy to conceive, that what is here done at 6 *per Cent.* may be done at any other Rate of Intereſt, by forming the Ratio, *viz.* r accordingly.

SCHOLIUM.

Altho' it be according to the Laws and Cuſtom of *England* to compute Intereſt at the Proportion of 6 *per Cent.* (as above;) yet he that takes up Money at Intereſt for any time leſs than even or compleat Years, pays more Intereſt than ſeems reaſonably due, according to the true Rules of Art.

As for Inſtance; If 100 *l.* be forborn at Intereſt one whole Year, it amounts to 106 *l.* But I ſay, if it be paid at the Half-year's end, it ſhould not amount to 103 *l.* as appears from this following Proportion:

Let a = the amount due at the Half-year's end; then it will be 100 .. a :: a .. 106, the Amount at the Year's end, *Ergo*, $aa = 10600$ and $a = \sqrt{10600} = 102.9563$ = 102 *l.* 19 *s.* 1½. Which is leſs than 103 *l.* by 10 *d.* ½. And if it be paid in leſs than Half a Year's time, the Error muſt needs be the greater.

Sect. 2. Of Annuities or Penſions in Arrears; Computed at Simple Intereſt.

Annuities or Penſions, &c. are ſaid to be in Arrears, when they are payable or due, either Yearly or Half-yearly, &c. and are unpaid for any number of Payments; therefore the Buſineſs is to compute what all thoſe Payments will amount unto, allowing any Rate of *Simple Intereſt* for their Forbearance, from the time each particular Payment became due: Now in order to that,

Put
$u =$ the Annuity, Penſion, or Yearly-Rent, &c.
$t =$ the time of its continuance (or being) unpaid.
$r =$ the Ratio, or Intereſt of 1 *l.* for one Year, as before.
$A =$ the Amount of the Annuity and its Intereſt.

Then, If $u =$ the firſt Year's Rent, due without Intereſt;
$2u =$ the Rent; $ru =$ the Intereſt } Due at the end of the ſecond Year;
$3u =$ the Rent; $ru + 2ru =$ the Intereſt } Due at the end of the third Year;
$4u =$ the Rent; $ru+2ru+3ru =$ the Intereſt } Due at the end of the fourth Year;
$5u =$ the Rent; $ru+2ru+3ru+4ru =$ the Intereſt } Due at the end of the fifth Year:

And ſo on for any Number of Years.

Hence 'tis evident, that $tu + \overline{ru+2ru+3ru+4ru+} \&c.$ continued to $t-1$ Terms is $= A$; that is (by Part 8, Chap. I, Theorem 3.)

$$\overline{t-1} \times ru + \frac{\overline{t-1}|^2 \times ru - \overline{t-1} \times ru}{2} \left(= \frac{ttru - tru}{2}\right) + tu = A.$$

Hence

Theorem 1. $\frac{\overline{tt-t}}{2} \times ru + tu = A = \frac{t^2u - tu}{2} \times r + tu.$

Theorem 2. $\frac{2A}{ttr - tr + 2t} = u$

Theorem 3. $\frac{2A - 2tu}{ttu - tu} = r.$

Theorem 4. $\overline{\frac{2A}{ru} + \frac{xx}{4}}\Big|^{\frac{1}{2}} - \frac{1}{2}x = t$; putting $\frac{2}{r} - 1 = x$

$tt - \frac{tru + 2tu}{ru} = \frac{2a}{ru}$ $\frac{2tu - tru}{ru} = \frac{2t}{r} - t =$ Queſt. $\frac{2}{r} - 1\ x\ t$

$tt - tx + \frac{xx}{4} = \frac{2a}{ru} + \frac{xx}{4}$ $x = \frac{2}{r} - 1$

$t - \frac{x}{2} = \sqrt{\frac{2a}{ru} + \frac{xx}{4}}$

Queſt. 1. If 250 *l.* Yearly Rent (or Penſion, &c.) be forborn or unpaid Seven Years; what will it amount to in that time at 6 *per Cent.* for each Payment as it becomes due?

Here is given $u = 250$, $t = 7$ and $r = 0.06$ to find A *per* Theorem 1.

First, $250 \times 7 = 1750 = tu$ and $1750 \times 7 = 12250 = ttu$.
Again, $12250 - 1750 = 10500 = ttu - tu$.

And $\frac{10500}{2} \times 0.06 = 315 = \frac{ttur - tur}{2}$

Laſtly, $315 + 1750 = 2065 = \frac{ttur - tur}{2} + tu = A$.

Viz. 2065 *l.* is the Anſwer required.

But if the Annuity, Rent, or Penſion, is to be paid by Quarterly or Half-yearly Payments, &c. then $\frac{0.06}{2} = 0.03 = r$ for Half-yearly Payments: And $\frac{0.06}{4} = 0.015 = r$ for Quarterly; or $0.045 = r$ for three Quarterly Payments.

Example of Half-yearly Payments.

Suppoſe 250 *l. per Annum*, to be paid by Half-yearly Payments, were in Arrears, or unpaid for Seven Years; what would it Amount to, allowing 6 *per Cent. per Annum* for each Payment as it became due?

In this *Example* there is given $u = 125 = \frac{250}{2}$, $t = 14$, the Number of Payments; and $r = 0.03 = \frac{0.06}{2}$ thence to find A.

First, $125 \times 14 = 1750 = tu$. $1750 \times 14 = 24500 = ttu$.
Again, $24500 - 1750 = 22750 = ttu - tu$.

Then $\frac{22750}{2} = 11375$.

And $11375 \times 0.03 = 341.25 = \frac{ttu - tu}{2} \times r$.

Laſtly, $341.25 + 1750 = 2091.25 = \frac{ttu - tu}{2} \times r + tu = A$

That is, A is $= 2091$ *l.* 5 *s.* 00 *d.* the Anſwer required.

N. B. Hence it may be obſerved, that Half-yearly Payments are more advantagious than Yearly.

For 2091 *l.* 5 *s.* > 2065 *l.* by 26 *l.* 5 *s.* 00 *d.* Conſequently Quarterly Payments are more advantagious than Half-yearly Payments.

Queſt. 2. What yearly Rent, Penſion, &c. being forborn or unpaid Seven Years, will raiſe a Stock of 2065 *l.* allowing 6 *per Cent. per Annum* for each Payment as it becomes due.

Here is given $A = 2065$, $t = 7$, and $r = .06$, to find u *per* Theorem 2.

Firſt, $7 \times .06 = .42 = tr$, and $.42 \times 7 = 2.94 = ttr$, then $ttr - tr = 2.52$, and $ttr - tr + 2t = 16.52$.

Laſtly, $16.52)\ 4130\ (250 = \frac{2A}{t^2r - tr + 2t} = u$

That is 250 *l. per Annum*, &c. will raiſe 2065 *l.* the Stock required.

Queſt. 3. In what time will 250 *l.* Yearly Rent, raiſe a Stock of 2065 *l.* allowing 6 *per Cent*, &c. for the forbearance of the Payments as they become due?

Here is given $u = 250$, $A = 2065$, and $r = 0.06$ to find t *per* Theorem 4.

Firſt, $\frac{2}{r} = \frac{2}{0.06} = 33.3333$ and $33.3333 - 1 = 32.3333 = x = \frac{2}{r} - 1$.

Then $16.1666 = \frac{1}{2}x$; and $261.3605 = \frac{1}{4}xx$.
Again $\frac{4130}{15} = 275.333 = 2A \div ru$.

And $275.3333 + 261.3605 = 536.6938 = \frac{2A}{ru} + \frac{1}{4}xx$; then $\surd 536.6938 = 23.1666$.

Laſtly, $23.1666 - 16.1666 = \sqrt{\frac{2A}{ru} + \frac{1}{4}xx} - \frac{1}{2}x = 7 = t$ the Time required.

Queſt. 4. If 250 *l.* yearly-Rent, being forborn ſeven Years, will amount to 2065 *l.* allowing Simple Intereſt for every Payment as it becomes due; what muſt the Rate of Intereſt be *per Cent.* &c.

Here is given $u = 250$, $A = 2065$ and $t = 7$ to find r *per* Theorem 3.

$$\text{Thus} \begin{cases} ttu = 12250 & 4130 = 2A \\ -tu = \underline{-1750} & \underline{3500} = \underline{2tu}. \end{cases}$$

$$ttu - tu = 10500)\quad 630 = 2A - 2tu\ (0.06 = r$$

Then $1 : 0.06 :: 100 : 6$ the Rate required.

Sect. 3. *The preſent Worth of Annuities or Penſions, &c. computed at Simple Intereſt.*

The Buſineſs of purchaſing Annuities or taking of Leaſes, *&c.* for any aſſigned time, depends upon the true equating of the Principal, or Money laid out on the Purchaſe, with the Annuity or Yearly-Rent, by allowing (or Diſcompting) the ſame Rate of Intereſt to both Parties: Which may be eaſily perform'd by duly applying the reſpective Theorems of the two laſt *Sections* together; as will fully appear by the following Queſtion.

Queſt. What is 75 *l.* Yearly-Rent, to continue Nine Years, worth in Ready-Money, at 6 *per Cent. per Annum* Simple Intereſt?

1. *Per* Theorem 1. of the laſt *Sect.* find what the propoſed Yearly-Rent wou'd amount to, if it were forborn Nine Years at 6 *per Cent.*

Thus, $u = 75$, $t = 9$ and $r = .06$. Quære A.

$$\left.\begin{array}{rcl} ttu & = & 6075 \\ tu & = & 675 \\ \hline ttu - tu & = & 5400 \end{array}\right\} \text{ then } 2)\,5400\,\left(\begin{array}{r} 2700 \\ .06 \end{array}\right\}\text{Multiply}$$

$$\begin{array}{rl} \hline 162 & = \dfrac{ttur - tur}{2} \\ 675 & = tu \\ \hline 837 & = A = \dfrac{t^2ur - tur}{2} + tu. \end{array}$$

2. Then by Theorem 2. *Sect.* 1. Find what Principal being put to Intereſt for the ſame time, and at the ſame Rate will amount to 837 *l.* $= A$

Thus, $tr = 9 \times 0.06 = .54$.

Therefore $tr + 1 = 1.54$

And $1.54)\,837\,(543.5064 = \dfrac{A}{tr+1} = p$

That is $p = 543$ *l.* 10 *s.* 1 *d.* ½ which is the Worth of 75 *l.* a Year, as was required.

From the Work of theſe two Operations (duly conſidered) it muſt needs be eaſy to conceive, how the two Theorems by which they were performed, may be combined into one.

For, 1. $\dfrac{ttru - tru + 2tu}{2} = A.$

And 2. $ptr + p = A.$

Conſequently $ptr + p = \frac{ttur - tur + 2tu}{2}$

And from this Equation may be deduced the following Theorems.

Theorem 1. $\left\{\frac{ttru - tru + 2tu}{2tr + 2} = p\right.$

Or $\left\{\frac{t^2r - tr + 2t}{2tr + 2} \times u = p\right.$

By this Theorem all Queſtions of the ſame kind with the laſt (*viz.* that above) may be eaſily and readily anſwered at one Operation.

Theorem 2. $\left\{\frac{2ptr + 2p}{ttr - tr + 2t} = u\right.$

Or $\left\{\frac{tr + 1}{t^2r - tr + 2t} \times 2p = u\right.$

Theorem 3. $\left\{\frac{2p - 2tu}{ttu - tu - 2pt} = t = \frac{2tu - 2p}{2pt + tu - ttu}\right.$

$ttur - tur - 2ptr + 2tu = 2$

Let $\left\{\frac{2}{r} - \frac{2p}{u} - 1 = x\right.$, then will $tt + \frac{2tu - tur - 2ptr}{ur} =$

$tt \pm xt = \frac{2p}{ru}$ Which gives $\frac{2tu - tur - 2ptr}{ur} = \frac{2t}{r} - 1 -$

$= \frac{2}{r} - 1 - \frac{2p}{u} \times t = tx$

Theorem 4. $\left\{\sqrt{\frac{2p}{ru} + \frac{xx}{4}} \pm \frac{x}{2} = t\right.$

By the Second and Fourth Theorems, two very uſeful Queſtions may be eaſily anſwered.

1. As for *Inſtance*: If it were required to find what Annuity or Yearly Rent, &c. may be purchas'd for any propoſed Sum, to continue any aſſigned Time, allowing any Rate of Intereſt.

This Queſtion may be anſwered by Theorem 2.

2. Again, If it were required to find how long any Yearly Rent, Penſion, or Annuity, &c. may be purchaſed (or enjoyed) for any propoſed Sum at any given Rate of Intereſt.

All Queſtions of this kind are eaſily anſwered *per* Theorem 4.

In theſe Queſtions it is ſuppoſed, that the Purchaſe, or Yearly Rent, is to commence, or be immediately entred upon. But if it be required to find the Value or Purchaſe of any Annuity or Yearly Rent, &c. in Reverſion; that is, when it is not to be Enter'd upon until after ſome time, or number of Years are paſs'd; then you muſt firſt find what the Sum propos'd to be laid out in the Purchaſe wou'd amount to if it were put to Intereſt, during the time the Annuity, &c. is not to be in preſent Poſſeſſion; and make that Amount the Sum for the Purchaſe, proceeding with it as in either of the two laſt Queſtions, &c.

Note, *From the firſt Queſtion of this Section, it will* [ſays my Author] *be eaſy to conceive how to perform the Equating of Payments between Debtor and Creditor, at any Rate of Intereſt, without doing any Damage to either Party.*

That is, when ſeveral Sums of Money are to be paid at ſeveral different Times, to find the time when all the Payments may be truly diſcharged at once: As if one Sum were to be paid at the end of two Months, another at ſix Months, and perhaps a third Sum at eight Months end, &c. And it were required to find the time when all thoſe Sums may be truly diſcharged at one Payment, without Loſs, &c.

CHAP. II.

Of Compound Intereſt *and* Annuities, *&c.*

COmpound Intereſt is that which ariſes from any Principal and its Intereſt put together, as the Intereſt ſtill becomes due; ſo that at every Payment, or at the time when the Payments became due, there is created a new Principal, and for that Reaſon it is called *Intereſt upon Intereſt*, or *Compound Intereſt*.

As for Inſtance; Suppoſe 100*l.* were Lent out for two Years, at 6 *per Cent, per Annum*, Compound Intereſt; then at the End of the firſt Year it will only amount to 106*l.* as in *Simple Intereſt*. But for the ſecond Year, this 106*l.* becomes Principal,

cipal, which will amount to 112 *l.* 7 *s.* 2 *d* ½ at the ſecond Year's end; whereas by *Simple Intereſt*, it would have amounted to but 112 *l.*

And altho' it be not Lawful to Let out Money at *Compound Intereſt*; yet in purchaſing of Annuities or Penſions, *&c.* and taking Leaſes in Reverſion, it is very uſual to allow *Compound Intereſt* to the Purchaſer for his Ready-money, and therefore it is very requiſite to underſtand it.

Sect. 1. Of Compound Intereſt.

Let { p = the Principal put to Intereſt, t = the time of its continuance, A = the Amount of the Principal and Intereſt, } as before.
R = { the Amount of 1 *l.* and its Intereſt for 1 Year, at any given Rate; which may be thus found,

Viz. 100 ·· 106 :: 1 ·· 1 . 06 the Amount of 1 *l.* at 6 *per Cent.*

Or, 100 ·· 107 :: 1 ·· 1 . 07 the Amount of 1 *l.* at 7 *per Cent.* And ſo on for any other aſſigned Rate of Intereſt.

Then, If R = the Amount of 1 *l.* for one Year at any Rate,
RR = the Amount of 1 *l.* for two Years,
R^3 = the Amount of 1 *l.* for three Years,
R^4 = the Amount of 1 *l.* for four Years,
R^5 = the Amount of 1 *l.* for five Years. Here $t = 5$.

For $1 \cdot\cdot R :: R \cdot\cdot R^2 :: R^2 \cdot\cdot R^3 :: R^3 \cdot\cdot R^4 :: R^4 \cdot\cdot R^5$, *&c.* in ÷

That is { As one Pound ·· is to the Amount of one Pound at one Year's end :: So is that Amount ·· to the Amount of one Pound at two Years End, *&c.*

Whence it is plain, that *Compound Intereſt* is grounded upon a Series of Terms Increaſing in Geometrical Proportion continued; wherein t (*viz.* the Number of Years) does always aſſign the Index of the laſt and higheſt Term, *viz.* the Power of R, which is R^t.

Again, $1 \cdot\cdot R^t :: p \cdot\cdot pR^t = A$ the Amount of p for the time that R^t = the Amount of 1 *l.*

That is { As one Pound ·· is to the Amount of 1 *l.* for any given Time :: So is any propoſed Principal (or Sum) ·· to its Amount for the ſame Time.

From the Premiſes the Reaſon of the following Theorems may be very eaſily underſtood.

2 years, which therefore must be $\frac{1}{r}$ or r^2, and is the Amount of £ for the first 2 years
manner, as 1 *l.* is to r its Amount for 1st year, so is r^2 to its Amount for 3 years, or Amount of 1 *l.*

Theorem 1. $pR^t = A$ as above.

From hence the two following Theorems are eaſily deduced.

Theorem 2. $\left\{ \frac{A}{R^t} = p \right.$

Theorem 3. $\left\{ \frac{A}{p} = R^t \right.$

By theſe three Theorems all Queſtions about *Compound Intereſt* may be truly Reſolved by the Pen only, *viz.* without Tables; tho' not ſo readily as by the help of Tables calculated on purpoſe.

But here I will ſhew how to ſolve ſuch Queſtions by the help of a Table of Logarithms: Thus;

$pR^t = A$, as before.

Therefore (by the Nature of *Logarithms*)

Theorem 1. $L,p + t \times L,R = L,A$

Theorem 2. $L,A - t \times L,R = L,p$

Theorem 3. $\frac{L,A - L,p}{t} = L,R$

Theorem 4. $\frac{L,A - L,p}{L,R} = t$

Queſt. 1. What will 375 *l.* 10 *s.* 00 *d.* amount to in Nine Years at 6 *per Cent. per Annum*, Compound Intereſt?

Here is given $p = 375.5$, $t = 9$, and $R = 1.06$ to find A *per* Theorem 1.

The *Logarithm* of 375 . 5 is = 2. 5746099
9 × *Logarithm* of 1 . 06 is = . 2277531

The Sum 2 . 8023630 = $L,p + t \times L,R$ = L,A; Therefore A is = 634 . 4 – = 634 *l.* 8 *s.* 00 *d. fere*, the Anſwer required.

Queſt. 2. What Principal (or Sum) muſt be put to Intereſt, to raiſe a Stock of 634 *l.* 8 *s.* 00 *d.* in Nine Years, at 6 *per Cent. per Annum*, &c?

Here is given $A = 634.4$, $R = 1.06$ and $t = 9$ to find p *per* Theorem 2. Thus;

The *Logarithm* of 634. is = 2.8023632.
9 × *Logarithm* of 1.06 is = .2277531

The Difference is = 2.5746101 =

$L, A - t \times L, R = L, p$; conſequently p is $= 375.5 +$; that is $p = 375$ *l.* 10 *s.* 00 *d.* + which is the Principal (or Sum) as was required.

Q. 3. In what time will 375 *l.* 10 *s.* 00 *d.* raiſe a Stock of (or amount to) 634 *l.* 8 *s.* 00 *d.* allowing 6 *per Cent. per Annum, Compound Intereſt?*

Here is given $A = 634.4$, $p = 375.5$ and $r = 1.06$ to find t by the fourth Theorem.

The *Logarithm* of 634.4 is = 2.8023632.
The *Logarithm* of 375.5 is = 2.5746099

The Difference is .2277533

And .2277533 ÷ 0.0253059 (the *Logarithm* of 1,6) is = 9 $t = \frac{L, A - L p}{L, R} = t$; that is, t is = 9 the number of Years required.

Queſt. 4. If 375 *l.* 10 *s.* 00 *d.* will amount to (or raiſe a Stock of) 634 *l.* 8 *s.* 00 *d.* in Nine Years; What muſt the Rate of Intereſt be *per Cent. per Annum?*

Here is given $A = 634.4$, $p = 375.5$ and $t = 9$.
Quere R *per* Theorem 3.

The *Logarithm* of 634.4 is = 2.8023632.
The *Logarithm* of 375.5 is = 2.5746099.

The Remainder is = .2277533;
Which Remainder Divided by 9, gives .0253059 =

$$\frac{L, A - L p}{t} = L, R.$$

And the Natural Number that anſwers the *Logarithm*, .0253059 will be found in the *Chiliads* to be 1.06
Then 1 : 1.06 − 1 :: 100 : 6 the Rate *per Cent.* required.

☞ *Note,* If the Logarithm of the Given Number, or if the required Number of the Given Logarithm be not exact enough for your purpoſe in your Table of Logarithms, you may make them as exact as you pleaſe by CHAP. 5. Part XV.

Now, I think, the Method of ſolving Queſtions *in* Compound Intereſt *by the foregoing manner (that is, by the help of* Logarithms) *preferrable to that of ſolving them by the help of* Tables; *For, to make a Table for every Rate of Intereſt that* may be, *or that* really is *Uſed, is too troubleſome.*

But * *my Author's Method is pretty good, which is thus.*

*Mr. Ward.

He makes a Table for the Amount of 1 *l.* at 6 *per Cent. per Annum*, Compound Intereſt, thus;

The Amount of 1 *l.* for 1 Year at the aforeſaid Rate, being 1 . 06 (R); the Amount of 1 *l.* for two Years, will be 1 . 1236 (R^2); the Amount of 1 *l.* for three Years, will be 1 . 191016 (R^3) *&c.*

And this Table he makes univerſally Uſeful for all Rates of *Compound Intereſt.* Thus;

Let x be = the difference between 1 . 06 = R the Amount of 1 *l.* for one Year in the *Table*, and any other propoſed Amount of 1 *l.* for one Year.

CASE 1. Then, if the propoſed Rate be Greater than 1 . 06 = R, $R+x$ will be = the true Amount of 1 *l.* for one Year at that Rate.

CASE 2. But if the propoſed Rate be leſs than 1 . 06 = R, then it will be $R-x$ = true Amount of 1 *l.* for one Year at that Rate.

Wherefore (by the 1st Theorem in this §. 1.) $p\times\overline{R+x}|^t$ is $=A$ in *Caſe* 1. And $p\times\overline{R-x}|^t = A$ in *Caſe* 2.

Wherefore, by Sir *Iſaac Newton*'s Theorem,

1. $$p\times\overline{R^t + tR^{t-1}x + t\times\frac{t-1}{2}\times R^{t-2}x^2 \pm t\times\frac{t-1}{2}\times\frac{t-2}{3}R^{t-3}x^3, \&c.} = A \text{ in Caſe} \begin{Bmatrix}1\\2\end{Bmatrix}$$

2. $$\frac{A}{R^t + tR^{t-1}x + t\times\frac{t-1}{2}R^{t-2}x^2, \&c.} = p$$

3. $$R^t + tR^{t-1}x + t\times\frac{t-1}{2}R^{t-2}x^2, \&c. = \frac{A}{p}$$

This Equation being Solv'd by the converging Series, the Value of x will thereby be found, and then $R+x$ or $R-x$ will ſhew the Rate of Intereſt.

And if the Time given or ſought be not terminated by whole Years, but by Weeks, Months, Quarters, or Half-years, &c. For Reſolving ſuch Queſtions, reduce thoſe parts of a Year into Days; that done, find an Anſwer according to the demand of the Queſtion (and agreeing to 1 *l.* as before) for thoſe number of Days; and in order to that, the Amount of 1 *l.* for one Day, at 6 *per Cent. per Annum* (as already found in the leſſer part of the *Logarithms*) will be very Uſeful; for by it a Table for the Amount of 1 *l.* for any number of Days at 6 *per Cent. per Annum,* may be calculated, thus;

The Amount of 1 *l.* for 1 Day at 6 *per Cent. per Annum,* being 1.0001596536, &c. The Amount of 1 *l.* for two Days, will be $1.0003193326, \&c. = \overline{1.0001596536, \&c.}|^2$; the Amount of 1 *l.* for three Days, will be $1.0004790372, \&c. = \overline{1.0001596536, \&c.}|^3$; &c.

Now, in order to make this Table of Amounts for Days, Uſeful for all Rates of Intereſt, you may proceed as before in that for Years.

Sect. 2. Of Annuities or Penſions *in Arrear, computed at* Compound Intereſt.

When Annuities, &c. are ſaid to be in Arrear; See Pag. 293. And I ſhall here make uſe of the ſame Letters to repreſent the ſame things as before in that Page, ſave only that R is here = the Amount of 1 *l.* for one Year, as in §. 1. of this Chapter.

Suppoſe u = the firſt Year's Rent of any Annuity without Intereſt,

Then will $Ru+u=$ { The Amount of the firſt Year's Rent and its Intereſt; more the ſecond Year's Rent, &c.

And $R^2u+Ru+u=$ { The Amount of the firſt and ſecond Years Rents with their Intereſt; more the third Year's Rent, &c.

Here $RRu + Ru + u = A$ the Amount of any Yearly Rent or Annuity, being forborn three Years; and from hence may be deduced theſe Proportions.

Viz. $u \cdot\cdot Ru :: Ru \cdot\cdot R^2u :: R^2u \cdot\cdot R^3u$, and ſo on in ∺ for any number of Terms or Years denoted by t, wherein the laſt Term will manifeſtly be $= uR^{t-1}$

Wherefore by *Part* 8. *Chap.* 2. *Prop.* 1.

$$A = \frac{uR^{t-1} \times R - u}{R - 1} = \frac{uR^t - u}{R - 1} \Big\} \text{ Theorem 1.}$$

$$\text{Hence Theorem 2.} \Big\{ \frac{RA - A}{R^t - 1} = u$$

$$\text{Theorem 3.} \Big\{ \frac{RA + u - A}{u} = R^t.$$

If this Equation be continually Divided by R until nothing remains, the number of thoſe Diviſions will be $= t$. *See Page* —— — Or rather from the Nature of *Logarithms.*

$$\text{Theorem 3.} \Big\{ \frac{L,\overline{RA - A + u} - L,u}{L,R} = t$$

$$\text{Theorem 4.} \Big\{ \frac{A}{u}R - R^t = \frac{A - u}{u}.$$

If this Equation be reſolved into Numbers, one of its Roots will ſhew the Value of R.

Queſt. 1. If 30 *l.* Yearly Rent or Annuity, &c. be forborn (*viz.* Remain unpaid) 9 Years; what will it amount to at 6 *per Cent. per Annum*, Compound Intereſt.

Here is given $u = 30$, $t = 9$, and $R = 1.06$, to find A *per* Theorem 1.

Firſt, In order to find what R^t is = to in this Queſtion, I proceed thus by the Logarithms.

The Logarithm of R^t = Logarithm of $\overline{1.06}|^{9} = 9 \times$ Logarithm of $1.06 = 9 \times .0253059 = .2277531$.

And the Number in the *Chiliads* that anſwers this Logarithm is 1.68948 *fere* $= R^t$
then $30 = u$

$$50.68440 - = uR^t$$
$$30 = u$$
$$20.68440 - = uR^t - u.$$

$R - 1 = .06$ And
$.06)\,20.68440\,(344.74 - =$

$$\frac{uR^t - u}{R - 1} = A = 344\,l.\ 14\,s.\ 9\,d.\ \tfrac{1}{2}$$ the Amount required.

Queſt. 2. What Yearly Rent or Annuity, &c. being Forborn or Unpaid 9 Years, will raiſe a Stock of 344*l.* 14*s.* 9*d.* ½ = 344.7395 at 6 *per Cent. per Annum*, &c.

Here is given $A = 344.7395$, $t = 9$, and $R = 1.06$ to find u by Theorem 2.

$$AR = 344.7395 \times 1.06 = 365.42387$$
$$A = 344.7395$$
$$RA - A = 020.68437$$

$$R^t - 1 = 1.689479 - 1. = .689479$$

$$.689479\,)\,20.68437\,(\,30 = \frac{RA - A}{R^t - 1} = u$$

Queſt. 3. In what time will 30*l.* Yearly-Rent raiſe a Stock of, or amount to 344 *l.* 14 *s.* 9 *d* ½. allowing 6 *per Cent. per Annum* for the forbearance of the Payments, &c.

Here is given $u = 30$, $A = 344.7395$, and $R = 1.06$; to find t *per* Theorem 3.

Firſt, $RA - A + u = 365.42387 - 344.7395 + 30 = 50.68437$.

And the Logarithm of 50.68437 is $= 1.7048743$
The Logarithm of $u = L,\ 30 = 1.4771212$

$$L,\ \overline{RA - A + u} - L,\ u = .2277531$$

$$LR = L;\ 1.06 = .0253059$$

$$.0253059)\,.2277531\,(9 = \frac{L,\overline{RA - A + u} - L,u}{L,R} = t;$$

that is, 9 Years is the time required.

Queſt. 4. If 30 *l.* *per Annum* being Unpaid Nine Years, will amount to 344 *l.* 14 *s.* 9 *d.* ½, allowing *Compound Intereſt* for every Payment as it becomes due; What muſt the Rate of Intereſt be *per Cent. per Annum*?

Here is given $u = 30$, $A = 344.7395$, and $t = 9$ to find R by Theorem 4.

Firſt, $\frac{A}{u} = \frac{344.7395}{30} = 11.491317$ and $\frac{A - u}{u} = 10.491317$.

Hence is this Equation $11.491317\,R - R^9 = 10.491317$; Wherefore R will be found by the converging Series or numeral Exegeſis, &c. $= 1.06$: and $1 \cdot\cdot 1.06 - 1 :: 100 \cdot\cdot 6$ the Rate *per Cent. per Annum* required.

Sect. 3. *To find the preſent Worth of* Annuities, Penſions, *or* Leaſes, &c. *at* Compound Intereſt.

Let $p =$ the preſent Worth of any Annuity, or Leaſe, &c. and the reſt of the Letters, as before:

Then from what has been ſaid in §. 3. *Chap.* 1. about Purchaſing of Annuities at *Simple Intereſt*, it will be eaſy to form the like Theorems here at *Compound Intereſt, viz.* by combining Theorem 1. *Page* 305; and Theorem 1. *Page* 299, into one Theorem.

For $\left\{ \frac{uR^t - u}{R - 1} = A \right\}$ The Amount of any Yearly Rent being unpaid any number of Years, *per* Theorem 1. of the laſt §.

And $pR^t = A$ $\Big\{$ The Amount of any Principal or Sum being put to Intereſt for the ſame number of Years, *per* Theorem 1. *Pag.* 299.

Hence it follows, that $pR^t = \frac{uR^t - u}{R - 1}$

Wherefore $pR^{t+1} - pR^t = uR^t - u$.

Or this Equation may be raiſed from the conſideration, that Purchaſing Annuities, or taking of Leaſes, &c. is grounded upon a Rank or Series of Geometrical Proportionals continually Decreaſing: Thus $\frac{u}{R}$ is the firſt and greateſt Term; R the common Ratio of all the Terms; and p is the Sum of all the Series;

That is, $\frac{u}{R} \cdot\cdot \frac{u}{RR} :: \frac{u}{RR} \cdot\cdot \frac{u}{R^3} :: \frac{u}{R^3} \cdot\cdot \frac{u}{R^4} :: \frac{u}{R^4} \cdot\cdot \frac{u}{R^5}$, &c.

in ∺ until the laſt Term $= \frac{u}{R^t}$, then (by *Part* 8. *Chap.* 2. *Prop.* 1.) will p be $= \frac{\frac{u}{R} \times R - \frac{u}{R^t}}{R - 1}$ Conſequently $pR^{t+1} - pR^t = uR^t - u$ as above.

From this Equation may be deduced theſe following Theorems.

Theorem 1. $\left\{ \frac{u - \frac{u}{R^t}}{R - 1} = p. \right.$

Theorem 2. $\left\{ \frac{pR^t \times R - pR^t}{R^t - 1} = u \right.$

Theorem 3. $\left\{ \frac{u}{p + u - pR} = R^t \right.$

Or from the Nature of Logarithms.

$\frac{L,u - L,\overline{p + u - pR}}{L,R} = t \Big\}$ Theorem 3.

Theorem 4. $\left\{ \frac{u}{p} = \frac{u}{p} R^t + R^t - R^{t+1} \right.$

The Reſolving of this Equation will diſcover the Value of R.

Queſt. 1. What is 30 *l.* Yearly-Rent to continue ſeven Years worth in ready Money; allowing 6 *per Cent.* Compound Intereſt to the Purchaſer.

Here is given $u = 30$, $t = 7$, and $R = 1.06$; to find p *per* Theorem 1.

$$\frac{u}{R^t} = \frac{30}{1.50363} = 19.9517 \;(\text{for } \overline{1.06}|^7 = 1.50363)$$

$$\text{And } 30 - 19.9517 = 10.0483 = u - \frac{u}{R^t}$$

Then $.06)\;10.0483\;(\;167.4716 = \dfrac{u - \dfrac{u}{R^t}}{R-1} = p =$ 167 *l.* 9 *s.* 5 *d.* Being the Anſwer required.

Queſt. 2. What Annuity or Yearly Rent, to continue ſeven Years, may be purchaſed for 167 *l.* 9 *s.* 5 *d.* allowing 6 *per Cent.* Compound Intereſt to the Purchaſer?

In this Queſtion, there is given $p = 167.4716$, $t = 7$; and $R = 1.06$; to find u by the ſecond Theorem.

Firſt, $pR^t \times R = 251.8153 \times 1.06 = 266.9242$

And $pR^t = 167.4716 \times 1.50363 = 251.8153$

$$pR^{t+1} - pR^t = 15.1089$$

Then $R^t - 1 = 0.50363$. and

$$0.50363\;)\;15.1089\;(\;30 = \frac{pR^t \times R - pR^t}{R^t - 1} = u;$$

that is, 30 *l.* is $= u$, the Anſwer required.

Queſt. 3. How long may one have a Leaſe of 30 *l.* Yearly Rent for 167 *l.* 9 *s.* 5 *d.* allowing 6 *per Cent. Compound Intereſt* to the Purchaſer?

Here is given $p = 167.4716$, $u = 30$ and $R = 1.06$, to find t by the third Theorem.

First $L, 30. =$ 1.4771212

Secondly, $p+u = 167.4716 + 30 = 197.4716$

And $pR = 177.5199$

$19.9517 = p+u-pR$

Then the Logarithm of $p+u-pR = L, 19.9517 =$ 1.2999799

$L, u - L, p+u-pR =$ 0.1771413

Now this Remainder Divided by .0253059 the Logarithm of $1.06 = LR$ gives 7 for a Quotient, and $= t$; that is, ſeven Years is the Anſwer required.

Queſt. 4. Suppoſe one ſhould give 167 *l.* 9 *s.* 5 *d.* for the Purchaſe of a Penſion, or Annuity of 30 *l. per Annum*, to continue ſeven Years; at what Rate of Intereſt *per Cent.* would that Purchaſe be made, allowing *Compound Intereſt* to the Purchaſer?

In this *Queſtion* there is given $p = 167.4716$, $u = 30$, and $t = 7$ to find R *per* Theorem 4.

The fourth Theorem is this Equation $\left\{ \frac{u}{p} = \frac{u}{p} R^{t} + R^{t} - R^{t+1} \right.$; which being brought into Numbers, and its Root Extracted, the Value of R will be found 1.06; *viz.* $R = 1.06$.

And then it will be 1 .. 0.06 :: 100 .. 6; the Rate *per Cent.* as was required.

These four Queſtions *include all the Varieties that can be propoſed about purchaſing Annuities or Leaſes,* &c. *which are either immediately to be enter'd upon, or in Poſſeſſion at the time when the Purchaſe is made.*

But ſuch Queſtions *as relate to Annuities, or taking of Leaſes,* &c. *in Reverſion, muſt be parted or divided into two diſtinct* Queſtions, *each to be ſeparately conſider'd by itſelf.* (See Page 298.) *As in the following Examples.*

Examp. 1. Suppoſe it were required to compute the preſent Worth of 75 *l.* Yearly Rent, which is not to commence, or be Enter'd upon, until Ten Years hence; and then to continue ſeven Years after that time, at 6 *per Cent,* &c. *Compound Intereſt.*

The firſt Work in this *Queſtion*, is to find what 75 *l. per Annum* to continue ſeven Years, is worth, Ready-money; as if it were immediately to be Enter'd upon: And to perform that, there is given $u = 75$, $R = 1.06$, and $t = 7$ to find p, as in the firſt *Queſtion* of this Section.

Thus $\frac{u}{R^t} = \frac{75}{1.50363} = 49.8793$ and $75 - 49.8793 =$

$25.1207 = u - \frac{u}{R^t}$

Now $R - 1 = 1.06 - 1 = .06$.

And 0.06) 25.1207 (418.6783 = 418 *l.* 13*s.* 6 *d.* $\frac{3}{4}$ the Anſwer to the firſt part of the Queſtion.

Then the next Work will be to find what Principal or Sum being Put out Ten Years at 6 *per Cent.* &c. will amount to 418 *l.* 13 *s.* 6 *d.* $\frac{3}{4}$.

Here is given $A = 418.6783$, $R = 1.06$, and $t = 10$ to find p per Theorem 2, *Page* 300.

Thus, $R^{10} = 1.790847$; and
1.790847) 418.6783 (233.7884 = 233 *l.* 15 *s.* 9 *d.*
the preſent Worth of 75 *l. per Annum* in Reverſion, *&c.* as was required.

Examp. 2. What Annuity or Yearly Rent to be entred upon Ten Years hence, and then to continue Seven Years, may be Purchas'd for 233 *l.* 15 *s.* 9 *d.* Ready-money at 6 *per Cent*, &c. *Compound Intereſt.*

In the firſt Work of this *Queſtion* there is given $p = 233.7884$, $R = 1.06$, and $t = 10$ (the time which the Annuity is not to be enter'd upon) to find A *per* Theorem 1. *Page* 300.

Thus $pR^t = 233.7884 \times 1.790847 = 418.6783 = A$ the Amount of 233 *l.* 15 *s.* 9 *d.* put to Intereſt Ten Years at 6 *per Cent.* &c.

Then for the ſecond Work of the *Queſtion*, there is given $p = 418.6783$, $R = 1.06$, and $t = 7$ (the time that the Annuity is to be enjoyed) to find u *per* Theorem 2. of this Section.

Thus $p \times R^t \times R = 418.6783 \times 1.50363 \times 1.06 =$

$$\begin{array}{rr} & 667.3095 \\ -pR^t = - & \underline{629.5372} \\ & 37.7723 = pR^{t+1} - pR^t \\ R^t - 1 = & 0.50363 \end{array}$$

.50363) 37.7723 (75 = u; that is u = 75 *l.* the Yearly Rent required by the Queſtion.

Theſe two Examples of finding p *and* u, *do fully ſhew the Method that muſt be uſed in Reſolving the two General, and indeed the moſt Uſeful Queſtions about Annuities or Leaſes in Reverſion: And if there be occaſion, either the Rate or the* Time; *viz.* R *or* t, *may be found by a due application of their reſpective* Theorems.

Note, If the Rents or Annuities, &c. are to be paid Half-yearly, or Quarterly, that

then $R = \overline{1.06}|^{\frac{1}{2}}$ for Half-yearly } Payments, at
and $R = \overline{1.06}|^{\frac{1}{4}}$ for Quarterly, &c. } 6 *per Cent.*

Thus far concerning ſuch Annuities *or* Leaſes, &c. *that are limited by any aſſigned Time; and 'tis only ſuch that can be computed by* Theorems *or certain* Rules. *However, it may not perhaps, be unacceptable to Inſert a Brief Account of ſome Eſtimates that have been reaſonably made by two Ingenious Perſons, about the Proportion or Difference of Men's Lives, according to their ſeveral Ages, which may be of good Uſe in computing the Values of Annuities, or taking Leaſes for Lives,* &c.

Sir William Petty *in his Diſcourſe made before the Royal Society* (Anno 1674.) *concerning the uſe of Duplicat Proportion, in the Life of Man and its Duration; ſaith, That it is found by Experience, there are more Perſons Living of between Sixteen and Twenty-ſix Years Old, than of any other Age or Decad of Years in the whole Life of Man* (viz. 70 *or* 80 *Years*). *His Reaſons for that Aſſertion I ſhall omit: But ſuppoſing it True, He thence infers, that the Roots of every Number of Men's Ages under Sixteen (whoſe Root is* 4) *compared with the ſaid Number* 4, *doth ſhew the Proportion of the likelihood of ſuch Men's reaching to the Age of* 70 *Years.*

As for Example; 'Tis 4 *times more likely, that one of* 16 *Years Old ſhould live to* 70, *than a New-born Babe; 'tis* 3 *times more likely, that one of* 9 *Years Old ſhould attain the Age of* 70, *than the ſaid Infant,* &c.

On the other Hand, 'tis 5 to 4, that one of Twenty-five Years Old will Die before one of Sixteen: And 6 to 5, that one of Thirty-ſix will die before one of Twenty-five: And ſo on, according to the Roots of any other declining Age, compared with (4, 58, &c.) *the Root of* 21, *which is the Year of Perfection according to the Senſe of our Law, and the Age for whoſe Life a Leaſe is moſt Valuable.*

2. *The Ingenious and great Mathematician, Captain* Edmund Halley, (*in* Philoſoph. Tranſact. *Numb.* 196.) *doth with great Induſtry and Skill, draw an Eſtimate of the Proportion of Men's Lives from the Monthly Tables of the Births and Funerals in* Breſlaw, *the Capital City of the Province of* Sileſia: *Whence He proves, that it's* 80 *to* 1, *a Perſon of Twenty-five Years Old will not Die in a Year: That 'tis* 5 ½ *to* 1, *That a Man of Forty will live Seven Years: That a Man of Thirty Years Old may reaſonably expect to live Twenty-ſeven, or* 28 *Years,* &c.

Now, from theſe and the like Proportions (he juſtly infers that) the Price of Inſurance upon Lives ought to be regulated, there being a great Difference between the Life of a Man of Twenty and one of Fifty; for Example, 'Tis 100 *to* 1, *that a Man of Twenty dies not in a Year, and but* 38 *to* 1, *for a Man of Fifty Years of Age. And upon theſe alſo depends the Valuation of Annuities for Lives: For it is plain, that the Purchaſer ought to pay only ſuch a part of the Value of any Annuity, as he hath Chances that he is Living. And for that purpoſe, he hath taken the pains (which was not a little) to compute the following Table, that ſhews the Value of Annuities for every 5th Year of Age to the 70th.*

Age.	*Year's Purchaſe.*	*Age.*	*Year's Purchaſe.*	*Age.*	*Year's Purch.*
1	10 . 28	25	12 . 27	50	9 . 21
5	13 . 40	30	11 . 72	55	8 . 51
10	13 . 44	35	11 . 12	60	7 . 60
15	13 . 33	40	10 . 57	65	6 . 54
20	12 . 78	45	9 . 91	70	5 . 32

Sect. 4. Of Purchaſing Freehold or Real Eſtates, at Compound Intereſt.

All *Freehold* or *Real* Eſtates, are ſuppoſed to be Purchaſed or Bought to continue for Ever, (*viz.* without any limited Time) therefore the Buſineſs of computing the true Value of ſuch Eſtates, is grounded upon a Rank or Series of Geometrical Proportionals continually Decreaſing *ad Infinitum*.

Thus, let p, u, R, denote the ſame *Data* as in the laſt *Section*;

then the Series will be $\frac{u}{R}, \frac{u}{R^2}, \frac{u}{R^3}, \frac{u}{R^4}, \frac{u}{R^5},$ and ſo on in ∺ until

the laſt Term $= 0$. then will $\frac{\frac{u}{R} \times R - 0}{R - 1} = p$ (By what has

been prov'd in ∺) $= \frac{u}{R-1}$.

This Equation affords theſe following Theorems.

Theorem 1. $p = \frac{u}{R-1}$ *ut Supra.*

Theorem 2. $pR - p = u$.

Theorem 3. $\frac{p+u}{p} = R$.

Example. Suppoſe a Freehold Eſtate of 75 *l.* Yearly Rent were to be Sold; What is it Worth, allowing the Buyer 6 *per Cent.* &c. *Compound Intereſt* for his Money?

In this Queſtion there is given $u = 75$, $R = 1.06$ to find p *per* Theorem 1. Thus;

$R - 1 = .06$) $75 = u$ ($1250 l. = p$ the Anſwer required.

And ſo for any of the reſt, as occaſion requires:

But if the Rent is to be paid either by Half-yearly or Quarterly Payments, then $R = \sqrt{1.56}$ for Half-yearly
And $R = \sqrt{:}\sqrt{1.06}$ for Quarterly
} Payments at 6 *per Cent.*

$$\text{Or}\left\{\begin{array}{l} R = 1.08 \text{ for Yearly} \\ R = \sqrt{1.08} \text{ for Half-yearly} \\ R = \sqrt{:}\sqrt{1.08} \text{ for Quarterly} \end{array}\right\} \begin{array}{l}\text{Payments} \\ \text{at 8 } \textit{per} \\ \textit{Cent.}\end{array}$$

The Like is to be underſtood for any other propoſed Rate of Intereſt either Greater or Leſs than 6 *per Cent.*

The Application of theſe Theorems *to Practiſe is ſo very Eaſy, that it's needleſs to Inſert more Examples.*

PART XVII.

Some of Diophantus's *Queſtions.*

Queſt. 1. TO find two Square Numbers Equal to a given Square Number $= dd$.

1. Take any two * Unequal Numbers; Suppoſe $s =$ the Greater, and $r =$ the Leſſer; then for the ſide of the firſt Square ſought, put ra.

2. And for the ſide of the ſecond Square ſought put $sa - d$ or $d - sa$.

3. Then from the firſt Step, the firſt Square is $= r^2 a^2$.

4. And from the ſecond Step, the ſecond Square is $= dd - 2dsa + ssaa$.

5. Therefore the Sum of the Squares ſought is $= r^2 a^2 + s^2 a^2 - 2dsa + dd$.

6. Which Sum muſt be Equal to the given Square $= dd$. Hence this Equation ariſeth $r^2 a^2 + s^2 a^2 - 2dsa + dd = dd$.

7. Which Equation after due Reduction, gives $a = \frac{2ds}{ss + rr}$

8. Therefore, by the firſt and ſeventh Steps, the ſide of the firſt Square ſought is now made known, for it is $= \frac{2rsd}{ss + rr}$.

9. And by the ſecond and ſeventh Steps the ſide of the ſecond Square is alſo made known, for it is

$$= \frac{ssd - rrd}{ss + rr}$$

Which two laſt Steps gives this.

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* *For if* s *was* $= r$ *then the ſecond Square would be* $= 0$, *and the firſt would be the given one.*

CANON.

Take any two Unequal Numbers, Multiply ſeverally the double of the Product of their Multiplication, and the Difference of their Squares by the ſide of the given Square, then divide thoſe Products ſeverally by the Sum of the Squares of the two Numbers firſt taken, and the Quotients ſhall be the ſides of the two Squares ſought.

Queſt. 2. To divide a Number given which is compos'd of two known Squares; Suppoſe $dd =$ the Greater, and $bb =$ the Leſſer, into two other Squares.

1. Take two unequal Numbers, s the Greater, and r the Leſſer, with this * Caution, *viz.* That s be not in ſuch Proportion to r as $d+b$ to $d-b$.

* *For if* s ·· r :: d + b ·· d − b *then the Squares that wou'd be found by the Canon, wou'd be the ſame with them given.*

2. For the ſide of the firſt of the two Squares ſought, put $ra+b$.

3. And for the ſide of the ſecond Square ſought, put $sa-d$, or $d-sa$.

4. Then the firſt Square ſought, is $=r^2a^2+2rba+bb$.

5. And the ſecond Square ſought is $s^2a^2-2sda+dd$.

Conſequently the Sum of thoſe Squares is $=r^2a^2+s^2a^2+2rba-2sda+b^2+d^2$.

7. Which Sum muſt be $=dd+bb$.

Hence the following Equation ariſeth, *Viz.*

$$\overline{s^2+r^2}\times a^2+\overline{2rb-2sd}\times a+b^2+d^2=b^2+d^2,$$

8. Which Equation after due Reduction, gives

$$a=\frac{2sd-2rb}{ss+rr}$$

9. Therefore from the eighth and ſecond Steps the ſide of the firſt Square ſought is now known and found

$$=\frac{2rsd+ssb-rrb}{ss+rr}$$

10. And from the eighth and third Steps the ſide of the ſecond Square ſought is likewiſe known and found,

$$=\frac{ssd-rrd-2rsb}{ss+rr}\text{ or }\frac{2rsb+rrd-ssd}{ss+rr}$$

That is, the former of thoſe two Quantities ſhall be the ſide of the ſecond Square, when $s^2 d - r^2 d$ is Greater than $2rsb$. But the latter of thoſe two Quantities ſhall be the Side when $s^2d - r^2d$ is Leſs than $2rsb$: Whence the ſide of the ſecond Square may be expreſs'd thus,

$$\frac{\overline{s^2 d - r^2 d,} \backsim 2rsb}{ss + rr}$$

From the Premiſes ariſeth the following

CANON.

Take any two Unequal Numbers, with this Caution, that the Greater may not have the ſame Proportion to the Leſſer, as the Sum of the ſides of the two Squares given hath to the Difference of the ſame Sides. Multiply the double Product of the Multiplication of thoſe two Numbers firſt taken by each of the ſaid two Sides given, and reſerve the Products: Multiply alſo the Difference of the Squares of the ſaid two Numbers firſt taken, by each of the ſaid two Sides given, and reſerve theſe Products. Then add the greater of the two firſt reſerv'd Products to the Leſſer of the two Latter, and reſerve the Sum for a Dividend. Take alſo the Difference between the Leſſer of the two firſt Products and the greater of the two Latter for a ſecond Dividend. Laſtly, Divide ſeverally the ſaid Dividends by the Sums of the Squares of the two Numbers firſt taken: So ſhall the Quotients be the ſides of the two Squares ſought.

Queſt. 3. To find two Square Numbers whoſe Difference ſhall be equal to a given Number: Suppoſe $= d$.

Solution.

1. Let ſome Number whoſe Square is leſs than the given Difference be repreſented by b.
2. For the ſide of the Leſſer Square ſought put a.
3. For the ſide of the Greater Square ſought put $a + b$.
4. Then the Leſſer Square is aa.
5. And the Greater Square is $a^2 + 2ab + bb$.
6. And the Difference of thoſe Squares is $2ab + b^2$.
7. But the ſaid Difference muſt be equal to the given Difference; therefore

$$2ab + bb = d$$

8. Which after due Reduction makes known the Value of the ſide of the leſſer Square, *Viz.* $a = \frac{d - bb}{2b}$

9. And from the 8th and 3d Steps the Value of the ſide of the Greater Square is alſo diſcovered, *Viz.* $a + b = \frac{d + bb}{2b}$.

CANON 1.

Take any Square Number leſs than the given Difference, and Subtract it from the ſaid Difference; then divide the Remainder by the double of the ſide of the Square firſt taken, and the Quotient ſhall be the ſide of the leſſer of the two Squares ſought. Laſtly, This ſide added to the ſide of the Square firſt taken, gives the ſide of the other Square ſought.

Again, Since by the two laſt Steps of the preceding Reſolution, the Values of the ſides of the two Squares ſought are, $\frac{d - bb}{2b}$ and $\frac{d + bb}{2b}$; Therefore, if we ſuppoſe $bc = d$; then thoſe Sides will be converted to theſe, *viz.* $\frac{bc - bb}{2b}$ and $\frac{bc + bb}{2b}$ Which laſt mention'd Sides after the common Factor b is caſt away, will be reduc'd to $\frac{1}{2}c - \frac{1}{2}b$, and $\frac{1}{2}c + \frac{1}{2}b$. Whence,

CANON 2.

Take two ſuch Unequal Numbers that the Product of their Multiplication may be equal to the given Difference, then half the Sum, and half the Difference of thoſe two Numbers ſhall be the ſides of the two Squares ſought.

Queſt. 4. To find two ſuch Square Numbers, that if to the Product of their Multiplication a given Number d be added, the Sum may be a Square.

Solution.

1. For one of the Squares ſought take any known Square Number, which may be repreſented by bb.
2. And for the other Square ſought put aa.
3. Then the Product of their Multiplication is $b^2 a^2$.
4. To which Product the given Number d being added, the Sum is, $b^2 a^2 + d$.

5. Which Sum muſt be equal to a Square, the ſide whereof may be ſuppos'd to be $ba -$ any known Number greater than $\sqrt{d}$; ſuppoſe $ba - c$, then the Square of $ba - c$, that is, $b^2a^2 - 2bca + c^2$ being equated to $b^2a^2 + d$, this Equation ariſeth, *viz.*

$$b^2a^2 + d = b^2a^2 - 2bca + cc.$$

6. Whence after due Reduction, $a = \dfrac{cc - d}{2bc}$.

From the Premiſes ariſeth this following

CANON.

For one of the Squares ſought take any Square Number, then from any Square Number ſubtract the given Number, and divide the Remainder by the double of the Product made by the Multiplication of the ſides of thoſe two Squares; ſo the Quotient ſhall be the ſide of the other Square ſought.

PART

PART XVIII.

Of the Alternations and Combinations of QUANTITIES.

CHAP. I.

Of the Alternations of Quantities.

DEFINITION.

Alternation, is a Word uſed by Mathematicians for the different Changes or Alternations of Order in any Number of Things propoſed taken one by one, two by two, or three by three, *&c.*

LEMMA.

The Number of Alternations of m Things $a^p b^q c^r$, &c. taken n by n, is equal to the number of Alternations of the $\overline{m-1}$ things a^{p-1} b^q c^r &c.

$+$ The Number of Alternations of $\overline{m-1}$ things $a^p b^{q-1} c^r$ &c.

$+$ The Number of the Alternations of the $\overline{m-1}$ things a^p b^q c^{r-1} &c.

$+$ *&c.*

taken $\overline{n-1}$ by $\overline{n-1}$.

Demonſtration.

It is evident, that by placing the Thing a in any determined Place; as, ſuppoſe, in the firſt place of every Alternation which can be made of the $\overline{m-1}$ things a^{p-1} b^q c^r, &c.

taken $\overline{n-1}$ by $\overline{n-1}$; that each Alternation by ſuch Poſition of a produc'd, will conſiſt of n things; and that all theſe Alternations in Number equal to the Number of Alternations of the ſaid $\overline{m-1}$ things $a^{p-1} b^{q} c^{r}$ &c. taken $\overline{n-1}$ by $\overline{n-1}$ are all the Alternations that can be made of the ſaid m things $a^{p} b^{q} c^{r}$ &c. taken n by n, which will have any a in the firſt Place of each Alternation of 'em.

For the ſame Reaſons the Number of Alternations of the ſaid m things $a^{p} b^{q} c^{r}$ &c. taken n by n, which will have b in the ſaid firſt Place of each of 'em, is equal to the Number of Alternations of the $\overline{m-1}$ things $a^{p} b^{q-1} c^{r}$ &c. taken $\overline{n-1}$ by $\overline{n-1}$.

Alſo the number of Alternations of the ſaid m things $a^{p} b^{q} c^{r}$ &c. taken n by n, which will have c in the ſaid firſt Place of each of them, is equal to the Number of Alternations of the $\overline{m-1}$ things $a^{p} b^{q} c^{r-1}$ &c. taken $\overline{n-1}$ by $\overline{n-1}$,

&c.

Wherefore the Number of Alternations of the m things $a^{p} b^{q} c^{r}$ &c. taken n by n, is equal to the Number of Alternations of the $\overline{m-1}$ things $a^{p-1} b^{q} c^{r}$ &c.

+ The Number of Alternations of the $\overline{m-1}$ things $a^{p} b^{q-1} c^{r}$ &c.

+ The Number of Alternations of the $\overline{m-1}$ things $a^{p} b^{q} c^{r-1}$ &c.

+ &c.

Taken $\overline{n-1}$ by $\overline{n-1}$. *Q. E. D.*

SCHOLIUM.

In order to find the Number of Alternations of m things $a^{p} b^{q} c^{r}$, &c. taken one by one, two by two, or three by three, &c. by the help of our *Lemma*: Let the number of the Indices in the ſaid m things, which are each either Equal to, or Greater than 1, 2, 3, 4, &c. be ſuppoſed $= A, B, C, D,$ &c.

reſpectively: Then the Number of the Indices in the ſaid m things, which are each equal to 1, 2, 3, &c. is $= \overline{A-B}$, $\overline{B-C}$, $\overline{C-D}$, &c. reſpectively. Then,

1. The Number of Alternations of the m things $a^p\,b^q\,c^r$ &c. taken one by one, is manifeſtly $= A$.

2. The Number of Alternations of m things $a^p\,b^q\,c^r$ &c. taken two by two, is by our *Lemma* $= \overline{A-B} \times$ Number of Alternations of $\overline{m-1}$ things, wherein $\overline{A-1}$ is equal to the Number of all the Indices,

$+\ B \times$ Number of Alternations of $\overline{m-1}$ things, wherein A is equal to the Number of all the Indices

Taken one by one.

$= \overline{A-B} \times \overline{A-1} + B \times A$, By *Parag.* 1. $= A \times \overline{A-1} + B$.

3. The Number of Alternations of m things $a^p\,b^q\,c^r$ &c. taken three by three, is, by our *Lemma*, $= \overline{A-B} \times$ Number of Alternations of $\overline{m-1}$ things, wherein $\overline{A-1}$, and B are equal to the Number of the Indices which are each either equal to, or greater than 1, and 2 reſpectively,

$+ \overline{B-C} \times$ Number of Alternations of $\overline{m-1}$ things, wherein A, and $\overline{B-1}$ are equal to the Number of the Indices which are each either equal to, or greater than 1, and 2 reſpectively,

$+ C \times$ Number of Alternations of $\overline{m-1}$ things, wherein A, and B are equal to the Number of the Indices, which are each either equal to, or greater than 1, and 2 reſpectively.

Taken two by two.

$= \overline{A-B} \times \overline{\overline{A-1} \times \overline{A-2} + B} + \overline{B-C} \times \overline{A \times \overline{A-1} + \overline{B-1}}$ $+ C \times \overline{A \times \overline{A-1}}\, B +$. By *Parag.* 2, $= A \times \overline{A-1} \times \overline{A-2}$ $+ 3AB - 3B + C$.

4. The Number of Alternations of m things $a^p\,b^q\,c^r$, &c. taken four by four, is, by our *Lemma*, $= \overline{A-B} \times$ Number

of Alternations of $\overline{m-1}$ things, wherein $\overline{A-1}$, B, and C are equal to the Number of the Indices which are each either equal to, or greater than 1, 2, and 3 Reſpectively,

$+\overline{B-C}\times$ Number of Alternations of $\overline{m-1}$ things, wherein A, $\overline{B-1}$, and C are equal to the Number of the Indices which are each either equal to, or greater than 1, 2 and 3, Reſpectively;

$+\overline{C-D}\times$ Number of Alternations of $\overline{m-1}$ things, wherein A, B, and $\overline{C-1}$ are equal to the number of Indices, which are each either equal to, or greater than 1, 2, and 3 Reſpectively,

$+D\times$ Number of Alternations of $\overline{m-1}$ things, wherein A, B and C, are equal to the Number of the Indices which are each either equal to, or greater than 1, 2, and 3 reſpectively; Taken three by three.

$$=\overline{A-B}\times\overline{\overline{A-1}\times\overline{A-2}\times\overline{A-3}+3B\times\overline{A-1}-3B+C}$$
$$+\overline{B-C}\times\overline{A\times\overline{A-1}\times\overline{A-2}+3A\times\overline{B-1}-3\times\overline{B-1}+C}$$
$$+\overline{C-D}\times\overline{A\times\overline{A-1}\times\overline{A-2}+3AB+3B+C-1}$$
$$+D\times\overline{A\times\overline{A-1}\times\overline{A-2}+3AB-3B+C};$$ By *Parag.* 3
$$=A\times\overline{A-1}\times\overline{A-2}\times\overline{A-3}+6A^2B-18AB+9B+3B^2+4CA-4C+D.$$

&c.

Example.

Let it be required to find the number of Alternations of $a^3b^3c^2$ (that is, of the 8 things *aaabbbcc*) taken four by four.

Here A is $=3$, $B=3$, $C=2$, $D=0$; Wherefore, by the 4th Parag. of our *Scholium*, $3\times2\times1\times0\ (0)+6\times3^2\times3\ (162)-18\times3\times3\ (-162)+9\times3\ (27)+3\times3^2\ (27)+4\times2\times3\ (24)-4\times2\ (-8)+0=70$; is the Number of Alternations of $a^3\ b^3\ c^2$ taken four by four.

COROLLARY.

From what has been ſaid in this *Scholium*, it is plain, that the Number of Alternations of m things different from each other, as $a\ b\ c\ d$, &c. taken n by n is (Becauſe in this Caſe, A is

equal

equal to m, and B, C, D, &c. are each equal to 0) $= A \times \overline{A-1} \times \overline{A-2} \times \overline{A-3} \times \overline{A-4} \times$ &c. continued to n places $= m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3} \times \overline{m-4} \times$ &c. continued to n Places.

Examples.

1. Let it be required to find the Number of Alternations of four Things different from each other, as *a b c d* taken four by four.

abcd,	*bacd,*	*cabd,*	*dabc,*
abdc,	*badc,*	*cadb,*	*dacb,*
acbd,	*bcad,*	*cbad,*	*dbac,*
acdb,	*bcda,*	*cbda,*	*dbca,*
adbc,	*bdac,*	*cdab,*	*dcab,*
adcb,	*bdca,*	*cdba,*	*dcba.*

Here $m = A$ is $= 4$, and $n = m = 4$, Wherefore by our *Coroll.* $4 \times \overline{4-1} \times \overline{4-2} \times \overline{4-3} = 24$ is the Number of Alternations of the four Things *a b c d* different from each other, taken four by four.

2. Let it be required to find the Number of Alternations of the five things *a b c d e* different from each other, taken two by two.

ab,	*ba,*	*ca,*	*da,*	*ea,*
ac,	*bc,*	*cb,*	*db,*	*eb,*
ad,	*bd,*	*cd,*	*dc,*	*ec,*
ae,	*be,*	*ce,*	*de,*	*ed.*

Here $m = A$ is $= 5$ and $n = 2$; Therefore, by our *Coroll.* $5 \times 4 = 20$ is the Number of Alternations of five things different from each other, taken two by two.

☞ *Note,* When n is $= m$, the Number of Alternations of m things $a^p\, b^q\, c^r$ &c. taken n by n; that is, m by m is $=$

$$\frac{m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3} \times \overline{m-4} \times \overline{m-5} \times \overline{m-6} \times \overline{m-7} \times \overline{m-8} \times \overline{m-9} \times \text{\&c continued to } m \text{ places.}}{p \times \overline{p-1} \times \overline{p-2} \times \overline{p-3} \times \text{\&c} \times q \times \overline{q-1} \times \overline{q-2} \times \overline{q-3} \times \text{\&c} \times r \times \overline{r-1} \times \overline{r-2} \times \overline{r-3} \times \text{\&c.} \times \text{\&c.}}$$

continued to p, q, r, &c. places respectively, which is a more regular and simple Series than them in the last *Parag.*

CHAP.

CHAP. II.

Of the Combination of QUANTITIES.

Definition.

THE ſeveral Ways, or different Caſes of taking any required Number of Things propos'd, without regarding their Order or Places, are called the *Combinations of Quantities.*

SCHOLIUM.

From the Nature of *Alternations* and *Combinations* rightly conſidered and compared, it will appear, That the Number of Alternations of m things $a\,b\,c\,d\,e$ &c. different from each other, taken n by n, is equal to the Number of *Combinations* of the ſaid m things taken n by n, multiplied by the Number of Alternations of n things different from each other, taken n by n: But the Number of Alternations of the ſaid m things taken n by n, is, by the *Coroll.* in the laſt Chap. $= m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3} \times$ &c. continued to n places; and the Number of Alternations of n things different from each other taken n by n, is, by the ſaid *Coroll.* $n = \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times$ &c. continued to n places; therefore the Number of Combinations of the ſaid m things taken n by n is $= \dfrac{m \times \overline{m-1} \times \overline{m-2} \times}{n \times \overline{n-1} \times \overline{n-2} \times}$ $\dfrac{\overline{m-3}}{\overline{n-3}} \times$ &c. Each Series continued to n Places or Terms.

Example.

Let it be required to find the Number of *Combinations* of the five things $a\,b\,c\,d\,e$ different from each other, taken three by three.

abc,	*acd,*	*ade,*	*bcd,*	*bde,*	*cde,*
abd,	*ace,*		*bce,*		
abe,					

Here m is $= 5$ and $n = 3$; Therefore $\dfrac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$ is the Anſwer required.

A

TREATISE

OF

Algebra.

BOOK II.

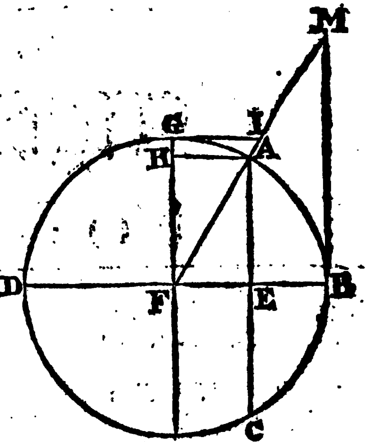
Note, Beſides the Characters already Explain'd in Book I. *Pages* 2, 3 and 4. theſe following are added, *Viz*.

r or R	Signifies	Radius
S		Sine
Σ		Co-Sine
ſ		Secant
σ		Co-Secant
T		Tangent
τ		Co-Tangent
U		Verſed-Sine
v		Verſed-Sine of the Supplement
$\angle$		Angle
∟		Right-Angle
$\perp$		Perpendicular
$\parallel$		Parallel
|		Right-Line
$\triangle$		Triangle
□		Square
▭		Oblong, Parallelogram, or Long-Square
⏢		Trapezium
ʒ		Sum
X		Difference
$\angle^s$		Angles.

PART I.

Of TRIGONOMETRY.

Definitions.



1. A Circle is ſuppoſed to be divided into 360 equal Parts, called Degrees, and each Degree into 60 equal Parts, called Minutes, and each Minute into 60 equal Parts called Seconds, *&c.* Any Portion of whoſe circumference is called an Arc, and is meaſured by the number of Degrees it contains.

2. A *Chord* is a Right-line joyning the extremities of an Arch, as A C is the Chord of the Arches A B C, A D C.

3. A *Right Sine* is a Right-line drawn from one End of an Arc, perpendicular to that Diameter paſſing through the other End; or it is half the Chord of twice the Arc. So A E is the Right-ſine of the Arcs A B, A D. And here it is Evident, that the Sine of 90 Degrees, (which is equal to the Radius or Semi-diameter of the Circle) is the greateſt of all Sines, the Sine of an Arc greater than a Quadrant, being leſs than the Radius.

N. B. *A great part of this Part* I. *was tranſcrib'd out of* Heyn's Trig.

4. A *Verſed Sine* is the Segment of the Diameter intercepted between the Arc and its Right-ſine, thus; E B is the Verſed Sine of the Arc A B, and E D of the Arc A D.

5. A *Tangent of an Arc* is a Right-line drawn perpendicular to the end of a Diameter paſſing through one end of an Arc, and its Length is limited by a Right-line drawn from the Center through the other End of the Arc, and this Line is called the *Secant*; thus, B M is the Tangent, and F M the Secant of the Arcs A B, A D.

6. The Difference of an Arc from a Quadrant, whether it be greater or leſs, is called its Compliment; ſo G A is the Compliment of the Arcs A B, A D; H A is the *Sine* of that Compliment, or Co-Sine; G I the Tangent of that Compliment, or Co-tangent; F I the Secant of that Compliment, or Co-ſecant.

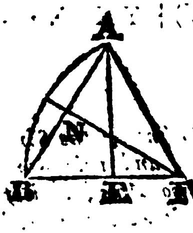
7. The Difference of an Arc from a Semi-circle is called its *Supplement*.

That part of the Radius which is betwixt the Center and Right-Sine, is equal to the Co-ſine; thus F E is = H A.

9. If an Arc be greater or leſs than a Quadrant, the Sum or Difference of the Radius and Co-ſine is equal to the Verſed Sine.

In a Triangle are Six Parts, *viz.* Three Sides and three Angles; any three of which being given (except the three Angles of a Plane Triangle) the other three may be found, either Mechanically by the help of a Scale of Equal Parts and Line of Chords, or by an Arithmetick Calculation (if ſuppoſing the Radius divided into any Number of Equal Parts, we know how many of thoſe Equal Parts are in the Sine, Tangent, or Secant of any Arc propoſed): The Art of inferring which is called *Trigonometry*, and is either Plain or Spherical.

The Table of Natural Signs, Tangents, &c. may be thus made.



1. A E the Sine of an Arc being given to find its Co-ſine F E.

☞ N. B. *The Radius is always ſuppos'd to be given*

$FAq: = AEq: + FEq:$ (By 47. 1. *Eucl. Elem.*)

$FAq: - AEq: = FEq:$ (By Tranſp.)

$\sqrt{FAq: - AEq:} = FE$ (By Evolution.)

That is, $\sqrt{Rq: - Sq:} = \Sigma$.

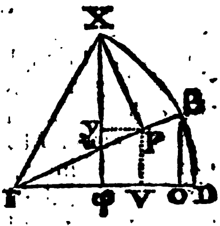
2. A E the Sine of an Arc being given, to find B N the Sine of half the Arc.

F E the Co-ſine, is known (by the 1ſt of this) and conſequently E B; then $\sqrt{AEq: + EBq:} = AB$ (by 47. 1. *Eucl. Elem.*) And $\frac{1}{2} AB = BN$ (by Third *Definition.*) *i. e.* $\frac{1}{2}\sqrt{Sq: + Uq:} = S\frac{1}{2}$ Arc.

3. BN the Sine of an Arc being given to find AE the Sine of twice that Arc.

FN the Co-ſine is (*or may be made*) known by the 1ſt; and the △s FBN and ABE have the ∠ N = ∠ E = ∟ and the ∠ B common to both, conſequently the ∠ NFB is = ∠ BAE (by 32. 1. *Eucl. Elem.*) Therefore the △s FBN and ABE are ſimilar; therefore (by 4. 6. *Eucl. El.*)

$$FB \cdot\cdot FN :: AB \cdot\cdot AE. \text{ i. e.}$$
$$R \cdot\cdot \Sigma :: 2S \cdot\cdot S\ 2\ \text{Arc.}$$



4. BO and XP, the Sines of two Arcs BD and XB, being given to find Xφ, the Sine of the Sum of the Arcs.

From the Point P let fall PV ⊥ DF and PY ⊥ Xφ.

Then the Co-ſines FO and FP are known (by the 1ſt.) And the △s FBO and FPV are ſimilar; therefore (by 4. 6. *Eucl. Elem.*) FB .. FP :: BO .. PV. Again, the △s XuP and Fuφ have the ∠s XPu and φ equal, as being each = ∟, and the ∠ PuX = ∠ φuF (by 15. 1. *Eucl. El.*) Wherefore they are Similar.

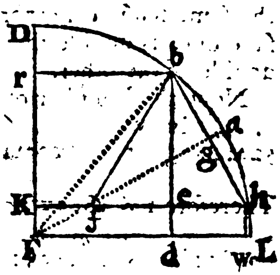
But the △s Fφu and FOB are alſo (manifeſtly) Similar; therefore the ∠ PXu is = ∠ BFO; and conſequently the △s PXY and BFO are Similar: Wherefore,

$$FB \cdot\cdot FO :: XP \cdot\cdot XY; \text{ But } Y\varphi\ (=PV) + YX = X\varphi = \frac{FO \times XP + FP \times BO}{FB};$$

That is, the Sine of one Arc into the others Co-ſine + the Sine of the other Arc into the Co-ſine of the firſt Arc, and this Sum divided by Radius, is = S, of the Sum of the two Arcs; and if the Sines of the ſame two Arcs were given, and it were required to find the Sine of their Difference, it would be

$$\frac{FO \times XP \backsim FP \times BO}{R} = S \textit{ Difference of the two Arcs.}$$

Let



5. Let the Arc La be 30 Degrees, and ab any Arc leſs than La, as alſo $ah = ab$. Draw bd and hw the Sines of the Arcs bL and hL, as alſo hK the Co-ſine of hL cutting bd in e: Draw bh, and the Radius aF cutting bh in g, and hK in f. Laſtly, draw br the Coſine of bL; then the $\llcorner^{d}$ Δs gbf and ebh having the $\angle b$ common are Similar; and conſequently the Angle hbe is $= gfb = LFa$, (by 29. 1. *Eucl. El.*) = 30 *deg.** therefore he is $= \frac{1}{2} bh = hg$ (*for the Sine of* 30 *deg. is* $= \frac{1}{2}$ *Radius*;) Therefore, by 47. 1. *Eucl. El.* $2ghq: - ghq: = beq: = 3ghq:$; conſequently $be = \sqrt{3} \times gh$; and $hw + gh \times \sqrt{3} = bd$; that is, the Sine of an Arc leſs 30 Deg. added to the Sine of its Defect $\times \sqrt{3}$ is = the Sine of an Arc as much exceeding 30 Degrees as the other wanted of 30 *deg.* So that if the Sines to 30 *deg.* be found, the reſt to 60° may be found by Additon and Multiplication into $\sqrt{3} = 1.732050807$, &c.

6. And ſince $eh = bg$; $rb + bg$ will be $= Kh$: that is the Sine of an Arc leſs than 60° added to the Sine of its Defect, gives the Sine of an Arc ſo much exceeding 60°.

Wherefore, after the Sines to 60° are found, the reſt to 90° may be found by Addition only. For Inſtance,

$S, 58° + S, 2° = S, 62°$
$S, 50° + S, 10° = S, 70°$.

An Application of what has beeen ſaid to the actual making the aforeſaid Table.

The Radius is Equal to the Chord of 60 Degrees, and $\frac{1}{2} R = S$, 30 Degrees, then by the 2d are known the Signs of the $\frac{1}{2}$ Arches.

Degrees.	′	″	‴	⁗	V
15	00	00	00	00	00
7	30	00	00	00	00
3	45	00	00	00	00
1	52	30	00	00	00
	56	15	00	00	00
	28	07	30	00	00
	14	03	45	00	00
	7	01	52	30	00
	3	30	56	15	00
	1	45	28	07	30
		52	44	03	45

* $\angle bfh$ is 60°, be
cauſe (by 4. 1. Euc
the $\llcorner^{d}$ Δ^{s} bfg, gf
are every way eq
therefore in Δ b
the $\angle^{s}$ fbh, fhb
each (by 32. 1. Eu
60°; and conſequ
ly $fh = bh$: So

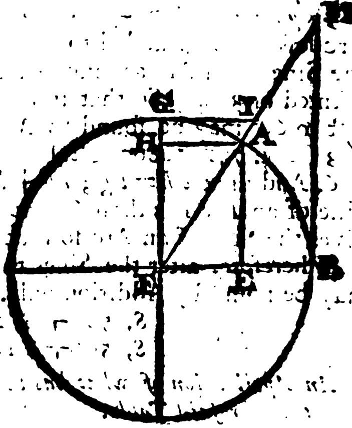
So that by twelve Diviſions we come to Sines, which have the ſame ſenſible Proportion as their Arches; for the laſt Sine ſave one, is double the laſt Sine to all Senſe, as one Arch is double of the other.

Now the Number of Minutes in 30 Degrees is 1800 and $52'':44''':03^{iv}:45^{v}$ is contained in 1800 Minutes, 2048 times; Therefore 1 × 1800 = 2048 × $52'':44''':03^{iv}:45^{v}$; therefore 1800 : 2048 :: Arc $52'':44''':03^{iv}:45^{v}$ .. Arc 1 :: S, $52'':44''':03^{iv}:45^{v}$ .. S, 1 *prope.*

And thus by a continued Biſection of Arches, the Sine of one Minute being had; by the 3*d* is found the Sine of 2 Minutes, by the 4th the Sine of 3 Minutes, and ſo on to 30 Degrees, then by the 5th to 60 Degrees, and by the 8th, the reſt to 90 Degrees.

Having found the Sines, the Tangents, Secants, *&c.* may be found by the following Proportions.

The △s FEA and FBM; as alſo the △s FHA and FGI are Similar; Therefore,



$$\left.\begin{array}{l} FE \,..\, FB :: EA \,..\, BM \\ FE \,..\, FB :: EA \,..\, FM \\ FH\,(EA) \,..\, HA\,(FE) :: FG \,..\, GI \end{array}\right\} \text{that is} \left\{\begin{array}{l} \Sigma \,..\, R :: S \,..\, T \\ \Sigma \,..\, R :: R \,..\, ſ \\ S \,..\, \Sigma :: R \,..\, \tau \end{array}\right\}$$

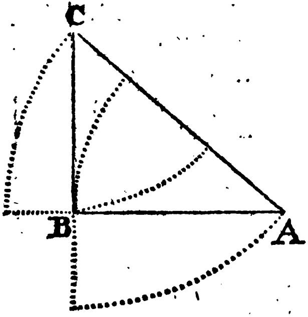
Plane and Spherical Trigonometry are each Solv'd by the help of four fundamental Propoſitions called *Axioms.*

CHAP. II.

CHAP. I.

Of Plane TRIGONOMETRY.

AXIOM I.



IN a $\llcorner^{d}$ $\triangle$ ABC. If one Leg of the $\llcorner$ as AB or CB be made the Radius of a Circle, then ſhall the other Leg CB or AB be the *Tangent* of the ∠ oppoſite to it, and the Hypothenuſe AC (or ſide oppoſite to the Right-angle) its *Secant* (by *Definition* 5.)

But if the Hypothenuſe AC be made the Radius of a Circle, then will the Legs (or Sides including the Right-angle) to wit, CB and AB be the Sines of the Angles oppoſite (by *Definition* 3.)

Upon this *Axiom* depends the Solution of the ſeven *Caſes* of $\llcorner^{d}$ Plane Triangles.

☞ *Note*, That the three Angles of a Plane Triangle make 2 $\llcorner^{s}$, or 180 Degrees. By 32. 1 *Eucl. El.*

For the more eaſy making the Proportion for the Solution of $\llcorner^{d}$ $\triangle$*s* obſerve, That as different Sides are made Radius, ſo the other ſides acquire different Names, which Names are either *Sines*, *Tangents*, or *Secants*, and are to be taken out of your Table.

To find a Side, any Side may be made Radius: Then ſay,

As the Name of the Side Given is to the name of the Side Required, ſo is the Side Given to the Side Required.

But to find an Angle, one of the given Sides muſt be made Radius; then,

As the Side made Radius is to the other Side, ſo is the Name of the firſt ſide (which is Radius) to the Name of the ſecond Side. Which fourth Proportional muſt be found among the *Sines* or *Tangents*, &c. to be determin'd by the Side made Radius, and againſt it is the Angle required.

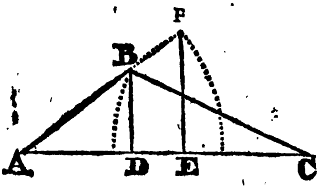
The

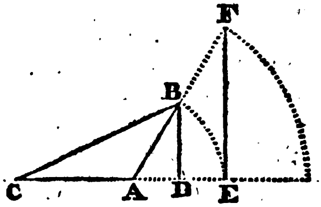
The Proportion for the Solution of the Seven Caſes of Plane Rightangled Triangles.

☞ *See the next foregoing Figure.*

Given.	*Required.*	*Propoſitions.*	*Radius.*	*Caſe.*
A B A and C	B C	Σ, A ·· S, A :: A B ·· B C R ·· T, A :: A B ·· B C τ A ·· R :: A B ·· B C	A C A B B C	1
A B A and C	A C	Σ, A ·· R :: A B ·· A C R ·· ſA :: A B ·· A C τ, A ·· σ, A :: A B ·· A C	A C A B B C	2
A B B C	A and C	A B ·· B C :: R ·· T, A Complement is C B C ·· A B :: R ·· T, C Complement is A	A B B C	3
A B B C	A C	A B ·· B C :: R ·· T, A then Σ A ·· R :: AB ·· A C Or $\sqrt{ABq:+BCq:}=AC$ (*per* 47. 1. *Euol. El.*	A B A C	4
A B A C	A and C	A C ·· A B :: R ·· Σ, A A B ·· A C :: R ·· ſ A	A C A B	5
A B A C	B C	A C ·· A B :: R ·· Σ, A Then R ·· T, A :: AB ·· B C Or $\sqrt{ACq:-ABq:}=BC$	A C A B	6
A C A and C	A B	R ·· Σ, A :: A C ·· A B ſ, A ·· R :: A C ·· A B σ, A ·· τ, A :: A C ·· A B	A C A B B C	7

AXIOM II.





In any Triangle the Sides are Proportional to the Sines of the oppoſite Angles.

Demonſtration.

Produce the leſſer ſide of the $\triangle ABC$, to wit, AB to F, making $AF = BC$; let fall the Perpendiculars BD, FE upon the ſide AC produc'd, if need be; then will FE be the Sine of the Angle A, and BD the Sine of the Angle C to the Radius $BC = AF$.

Now the $\triangle s$ ABD and AFE having the $\angle A$ common to both, and the $\angle D = \angle E = \llcorner$, are Similar; wherefore (by the 4. 6. *Eucl. El.*)

$AF(BC) \cdot\cdot AB :: FE \cdot\cdot BD$; that is
$:: S,A \cdot\cdot S,C.$ Q. E. D.

Otherwiſe thus,

By Axiom 1. $\left\{\begin{array}{l} AB \cdot\cdot R :: BD \cdot\cdot S,A \\ \qquad\text{and} \\ BC \cdot\cdot R :: BD \cdot\cdot S,C: \end{array}\right.$

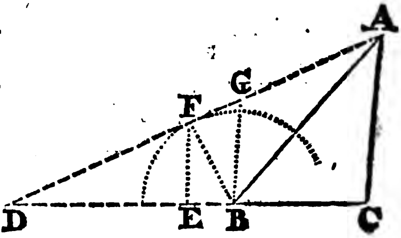
Therefore $AB \times S,A \,(= R \times BD) =$
$BC \times S,C$;

Wherefore $AB \cdot\cdot BC :: S,C \cdot\cdot S,A.$ Q. E. D.

AXIOM 3.

A X I O M. III.

The Sum of the Legs of an Angle is to their Difference as the Tangent of half the Sum of the Angles oppoſite to thoſe Legs, is to the Tangent of half their Difference.



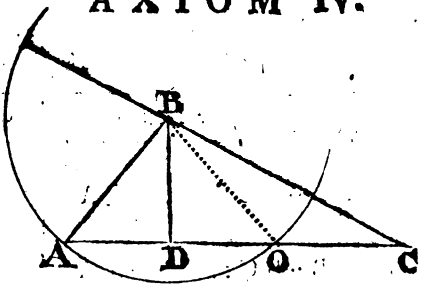
Demonſtration.

In the △ A B C produce C B one of the given Legs of the Angle given, till B D becomes = B A, and then biſect C D in E, join A D and biſect it alſo in F; draw B F, which (by 8. 1. *Eucl. El.*) will be ⊥ A D, and draw E F, which (by 2. 6. *Eucl. El.*) will be ∥ A C. Then will the Angle A B F = F B D = $\frac{1}{2}$ A B D, which external Angle A B D is (by 32. 1. *Eucl. El.*) = B A C + C; that is, to the Sum of the oppoſite Angles required.

Draw then B G Parallel to C A, ſo will the Angle G B A be (by 29. 1. *Eucl. El.*) equal to its alternate one B A C; and if from half the Sum of the oppoſite Angles you take the leſſer Angle *i.e.* If from ∠ ABF you take ∠ G B A, there will remain ∠ GBF = half the difference of the oppoſite Angles: And ſo alſo if from C E half the Sum of the Legs, you take C B the leſſer Leg, there will remain B E equal to half the difference of the Legs: And then, ſince the △ A B F is Right-angled, if B F be made Radius, A F will be the Tangent of ∠ A B F (*i. e.* the Tangent of half the Sum of the oppoſite Angles.) But the Segments of the Legs of any Triangle cut by Lines Parallel to the Baſe being (by 2. 6. *Eucl. El.*) proportional; E C ·· E B ∷ F A ·· F G; that is in Words, half the Sum of the Legs, is to half their difference, as the Tangent of half the Sum of the oppoſite Angles, is to the Tangent of half their difference; *For* $\frac{1}{2}$ *Sum* + $\frac{1}{2}$ *difference of any two Quantities is = greater of them. And* $\frac{1}{2}$ *Sum* — $\frac{1}{2}$ *diff. is = leſſer of them.*

But Wholes are as their Halves: Wherefore the Sum of the Legs is to their Difference as the Tangent of half the Sum of the ∠s oppoſite is to the Tangent of half their difference. *Q. E. D.*

AXIOM IV.



Having drawn a ⊥ B D from an Angle to its oppoſite Baſe; A C ſhall be to the Sum of the other two ſides A B and B C, as their difference B C — A B is to the difference of the Segments of the Baſe O C.

Demonſtration.

For by the 35. 3 *Eucl. El.* $\overline{BC-BA}\times\overline{BC+BA}$ is $=OC\times AC$; therefore $AC \cdot\cdot BC+BA :: BC-BA \cdot\cdot OC$. *Q. E. D.*

◆

The Proportions for the Solutions of the Six Caſes of Plane Oblique Triangles.

Given.	*Required.*	*Proportions.*	*Axioms*	*Caſe.*
A B B C and C	A	A B ·· B C :: S, C ·· S, A	2	1
A B B C and C	A C	A B ·· B C :: S, C ·· S, A Hence by Subtraction the ∠ B will be known. S, A ·· S, B :: B C ·· A C	2	2
A, C & BC	A B	S, A ·· S, C :: B C ·· A B	2	3
B A B B C	A and C	$BC + AB \cdot\cdot BC - AB ::$ $T\frac{1}{2}Z\angle^{s}$ oppoſite ·· $T\frac{1}{2}X$ Angles oppoſite. Then, $\frac{1}{2}Z + \frac{1}{2}X =$ Greater ∠, A $\frac{1}{2}Z - \frac{1}{2}X =$ Leſſer ∠, C	3	4
B A B B C	A C	Firſt find the Angles by the laſt; Then S, C ·· S, B :: A B ·· A C	3 and 2	5
A B B C A C	A B C	AC ·· BC+BA :: BC−BA ·· OC Then $\frac{1}{2}$ AC + $\frac{1}{2}$ OC = DC And $\frac{1}{2}$ AC − $\frac{1}{2}$ OC = DA Then A B ·· A D :: R ·· Σ, A And C B ·· D C :: R ·· Σ, C And B C ·· A C :: S, A ·· S, B Or, B A ·· A C :: S, C ·· S, B	4 1 and 2	6

CHAP. II.

CHAP. II.

Of *Spheric* TRIGONOMETRY.

A *Spherick Triangle* is that which is contain'd between the Arches of three great Circles of the Sphere.

A *Spherick Angle* is the ſame with the Inclination of the Planes of thoſe two Circles which conſtitute the Angle.

Affections of Spherick Triangles.

1. When a Circle falls on another, the Sum of the two Angles made thereby is $= 2\,\llcorner$.

2. When a Circle croſſes another Circle, the Vertical Angles made thereby are mutually Equal.

3. The greater Angle is oppoſite to the greater Side.

4. An *Iſoſceles Triangle* hath its two Angles at the Baſe mutually Equal; and on the contrary, If a *Triangle* has two Angles Equal, it has two Sides Equal.

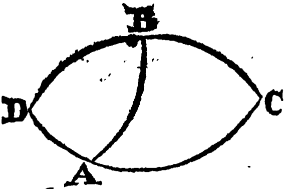
5. Two *Triangles* mutually Equilateral are alſo Equiangular one to another.

6. If there be two Triangles, and in each one Angle, and the two Sides including that Angle reſpectively Equal; or if one ſide and the two Angles adjacent be ſeverally equal; then the two *Triangles* are equal, for if laid one upon another they will agree.

7. Two ſides of a *Triangle* are bigger than one; for the Arch of a Great Circle is the ſhorteſt Diſtance between two Points on the ſurface of a Sphere, as a Streight line is betwixt two Points in a Plane.

Theſe ſeven Properties are common to Plane *as well as* Spheric Triangles, *and have a like Demonſtration.*

8. All Great Circles mutually cut each other into two equal Parts; for the common Section of their Planes is a Diameter of the Sphere, and conſequently the two Sections of the *Peripheries* are at a Semicircle Diſtance.



Hence it follows, That every ſide of a *Spheric Triangle* is leſs than a Semi-circle; thus B D is leſs than the Semi-circle D C.

9. The oppoſite Angles at the Sections of the two Circles are Equal; thus ∠ D is = ∠ C, for the ſame Planes make both Angles.

10. In any *Spherick Triangle*, if the Sum of the Legs of an Angle, as D B + B A be $\left.\begin{matrix}\sqsubset\\=\\\sqsupset\end{matrix}\right\}$ a Semi-circle D B C, the intern Angle at the Baſe, *viz.* D, is accordingly $\left.\begin{matrix}\sqsubset\\=\\\sqsupset\end{matrix}\right\}$ the outward oppoſite Angle BAC; and conſequently the Sum of the two internal Angles at the Baſe, towit D + D A B ſhall be $\left.\begin{matrix}\sqsubset\\=\\\sqsupset\end{matrix}\right\}$ two Right-angles.

Demonſtration. nor DB + BC

(a) Axiom 1.
(b) 9 *hujus.*
(c) 3 *hujus.*

If *D B* + *B A* be $\left.\begin{matrix}\sqsubset\\=\\\sqsupset\end{matrix}\right\}$ *D B C*, then *B A* is (a) $\left.\begin{matrix}\sqsubset\\=\\\sqsupset\end{matrix}\right\}$ *B C*; therefore the ∠*C* or (b) which is Equal to it ∠ D is $\left.\begin{matrix}\sqsubset\\=\\\sqsupset\end{matrix}\right\}$ *B A C*, and ∠*D* + ∠ *D A B* is $\left.\begin{matrix}\sqsubset\\=\\\sqsupset\end{matrix}\right\}$ (∠ *B A C* + ∠ *D A B* =) 2 ∟.

COROLL.

11. In an *Iſoſceles Spheric Triangle*, if one of the equal Legs is $\left.\begin{matrix}\sqsubset\\=\\\sqsupset\end{matrix}\right\}$ a Quadrant, the Angle at the Baſe is $\left.\begin{matrix}\sqsubset\\=\\\sqsupset\end{matrix}\right\}$ ∟.

[illegible]

The

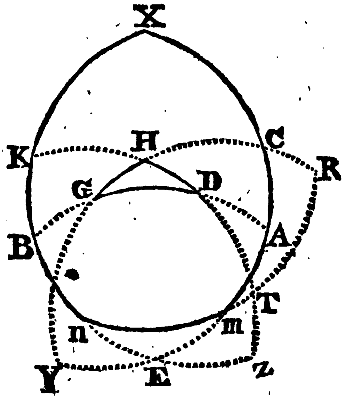
12. The Sum of the three ſides of a *Spheric Triangle* is leſs than a Circle, for *BA $(^{e}) \sqsupset BC + AC$; therefore $DB + DA + BA\,(^{a}) \sqsupset DBC + DAC$.

* *See the preceding Fig.*

$(^{e})$ 7 *hujus.*

13: If from the point of an Angle as a Pole you deſcribe a Great Circle, or which is the ſame, if you deſcribe a Circle at the diſtance of 90 Degrees from the ſaid Point, the Arc of this Circle intercepted between the Legs of the Angle, is the meaſure of the Angle.

14. The Poles of the ſides of any *Spheric Triangle* GHD conſtitute another Triangle *n* X *m* which we may call Supplemental to the Triangle GHD; for the Supplements of the $\left\{\begin{matrix}\textit{Angles}\\ \textit{Sides}\end{matrix}\right\}$ of the Triangle *n*X*m* are equal to the $\left\{\begin{matrix}\textit{Sides}\\ \textit{Angles}\end{matrix}\right\}$ of the Triangle GHD [illegible]



Demonſtration.

From the Points G, H, D as Poles, deſcribe on the Globe three Great Circles XAY, R*m*T*n*, XB*n*Z; then *m* being the Pole of GH, mY is therefore equal to a Quadrant = AX, X, or E being the Pole of GD; therefore, adding *m*A on both ſides, we have *m*X = AY = ſupplement of AC, the meaſure of the Angle HGD.

Again, Z*n* is equal to a Quadrant (*n* being the Pole of HD) = BX, and adding B*n* on both ſides, *n*X is = BZ = Supplement of BK the meaſure of the ∠ HDG.

Alſo *n*T is = a Quadrant = *m*R, and ſubtracting *m*T on both ſides, *nm* is = TR the meaſure of the Supplement of ∠ GHD.

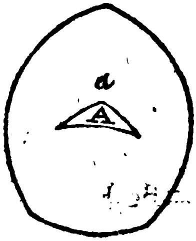
And here obſerve, that the *Triangle n* E *m* conſtituted between the three next Poles, hath ſeverally its three $\left\{\begin{matrix}\textit{Sides}\\ \textit{Angles}\end{matrix}\right\} = \left\{\begin{matrix}\textit{Angles}\\ \textit{Sides}\end{matrix}\right\}$ of the *Triangle* GHD, ſave that the Greateſt ſide *nm* is the Supplement of the Angle GHD, and the Angle *n*E*m* of the ſide GD.

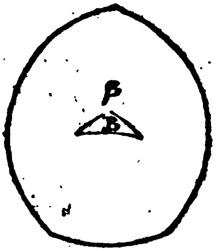
^ DB+DA+BC+AC or DB+BC+
DA+AC or

Any

15. Any Angle of a *Triangle* added to the difference of the other two is $\sqsupset 2\llcorner$ for * $Xn \sqsupset Xm + mn$; *i.e.* $2\llcorner - HDG \sqsupset 2\llcorner - HGD + 2\llcorner - GHD$, and taking away $2\llcorner$ from each ſide $-HDG \sqsupset 2\llcorner - HGD - GHD$, and adding $HGD + GHD$ on both ſides, we have $HGD + GHD - HDG \sqsupset 2\llcorner$.

* *See the foregoing Fig.*





16. Two *Spheric Triangles* A and B mutually Equiangular, are alſo mutually Equilateral; for becauſe they are Equiangular, their Supplemental *Triangles* α and β are Equilateral by the 14°, and therefore Equiangular by the 5°, and therefore the propos'd *Triangles* A and B, whoſe ſides are the Supplements of thoſe Angles, are Equilateral by the 14°.

17. The Sum of the three Angles of any *Spheric Triangle* is $\sqsubset 2\llcorner$ and $\sqsupset 6\llcorner$ * for $Xn + Xm + nm \sqsupset 4\llcorner$ (by the 12th); that is $2\llcorner - HDG + 2\llcorner - HGD + 2\llcorner - GHD \sqsupset 4\llcorner$; that is $6\llcorner - HDG - HGD - GHD \sqsupset 4\llcorner$, and taking $4\llcorner$ from each ſide, there remains $2\llcorner - HDG - HGD - GHD \sqsupset 0$, and by adding $HDG + HGD + GHD$ on both ſides $2\llcorner \sqsupset HDG + HGD + GHD$.

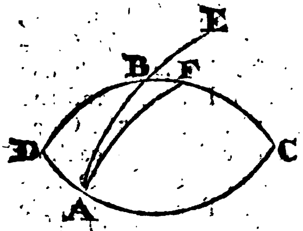
* *Vid. Fig. in 14 hujus.*

2*dly*, The Sum of the *Intern Angles* is leſs than the Sum of the *Intern* and *Extern*, both which in all make but $6\llcorner s$.

Or thus (by the *Fig.* in the 14th.) the Supplement of nm is the Meaſure of GHD; $Em =$ the Meaſure of HGD, and En of HDG: But the Supplement of nm added to nm makes $2\llcorner$, and $En + Em$ is * $\sqsubset nm$; therefore the Supplement of nm added to $En + Em$ is $\sqsubset 2\llcorner$.

(*) 7. *hujus.*

Or thus,



The $\angle$ B A C is $\left.\begin{matrix}\sqsubset\\=\\\sqsupset\end{matrix}\right\}$ $\angle$ B D A.

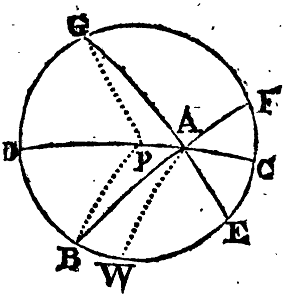
In this firſt Caſe make F A C = B D A, and produce A B to E.

Then D F + A F is $^{(f)}$ = 2 $\llcorner$; * therefore B F + A F $\sqsupset$ 2 $\llcorner$; therefore $\angle$ F B E (= D B A) $^{(f)}$ $\sqsubset$ F A B; But F A D + F D A $^{(f)}$ = 2 $\llcorner$: *i. e.* F A B + B A D + B D A = 2 $\llcorner$; therefore D B A + B A D + B D A $\sqsubset$ 2 $\llcorner$. (= $\angle$ C, and so B A = B C) (f) 10 *hujus.*

2dly, If $\angle$ B A C = $\angle$ D, then BAD + BDA $^{(f)}$ = 2 $\llcorner$; therefore $\angle$s B A D + BDA + A B D $\sqsubset$ 2 $\llcorner$.

3dly, If $\angle$ B A C $\sqsupset$ $\angle$ D, then $\angle$s B A D + B D A $^{(f)}$ $\sqsubset$ 2 $\llcorner$; therefore $\angle$s B A D + B D A + A B D $\sqsubset$ 2 $\llcorner$.

18. Of ſeveral Arcs of Great Circles falling from the ſame Point of the Sphere's ſurface on another Circle; the greateſt is that which paſſes through the Pole of the Circle, and the next to this is greater than that which is farther off: For ſuppoſe P the Pole of the Circle C w D, and w the Pole of D P C, then is A D (= A P + P B) $\sqsubset$ $^{(e)}$ A B $\sqsubset$ A E $\sqsubset$ A C and the Arc B w C $\sqsubset$ B P $\sqsubset$ B D.



19. A Great Circle paſſing through the Poles of another Great Circle cuts it at Rightangles; and on the contrary, if it cuts it at Rightangles, it paſſes through its Poles; for the Angle of the Inclination of the Planes is then Right; thus, P B D = $\llcorner$ = P G D = P D B = w A C. *See the foregoing Fig.*

* because (by 9) FAC is = C, and so (by 4.) FC is = AF, and so DF + FC = 2 $\llcorner$ is = DF + AF.

20. In an Oblique-angled *Spheric Triangle* if the ∠s at the Baſe are like, that is, both acute, or both obtuſe, the Perpendicular let fall from the oppoſite Angle falls within the *Triangle*, and the Quadrantal Arc without: But if they are unlike, the Perpendicular falls without, and the Quadrant within; for the * *Triangle* E A F has each of the Angles E and F acute, and the Perpendicular A C falls within, and the Quadrant A w without: Alſo the * Triangle B A G has each of the ∠s B and G obtuſe, and the Perpendicular A D falls within, and the Quadrant A w without. But the Triangle B A E has the ∠s B and E of different kinds, and the Perpendicular A C without, and the Quadrant A w within.

* *See the foregoing Fig.*

By the ſame *Figure* may the Ambiguities of ∟^{d} *Spheric Triangles* be Solved.

Solutions.

1. The Legs of the Right-angle are of the ſame Kind with the oppoſite Angles: So in the Triangle B D A, becauſe D A is $\sqsubset$ Quadrant D P, the ∠ D B A is $\sqsubset$ ∟ D B P; and in the *Triangle* B C A, becauſe A C is $\sqsupset$ Quadrant P C, the ∠ C B A is $\sqsupset$ ∟ C B P.

2. If the Legs (and conſequently the Angles) are of the ſame, or of different Kind, the Hypothenuſe is accordingly Leſs or Greater than a Quadrant: So in the *Triangles* E D A and E C A, the * Hypothenuſe A E is $\sqsupset$ Quadrant; but in the *Triangle* B D A the Hypothenuſe A B is (e) $\sqsubset$ Quadrant, B P.

* *Side oppoſite.*

3. *Vice verſa.* If the Hypothenuſe be $\sqsupset$ 90 Degrees, the Legs are Similar, and the two oblique Angles are Similar. But if the Hypothenuſe be $\sqsubset$ Degrees, 90 the Legs are diſſimilar, and the Oblique Angles diſſimilar.

4. If the Hypothenuſe be $\left.\begin{matrix}\sqsupset\\ \sqsubset\end{matrix}\right\}$ 90 Degrees, each Leg ſhall be $\left\{\begin{matrix}\textit{Similar}\\ \textit{Dissimilar}\end{matrix}\right\}$ to its adjacent Angle: This follows from the firſt and ſecond *Solutions.*

5. And on the contrary, if a Leg be $\left\{\begin{matrix}\textit{Similar}\\ \textit{Dissimilar}\end{matrix}\right\}$ to its adjacent Angle, the Hypothenuſe will be $\left.\begin{matrix}\sqsupset\\ \sqsubset\end{matrix}\right\}$ 90 Degrees.

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Fig.)
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DF; which
Wherefore,

By 47. 1. *Eucl. El.* $PFq - PDq = DFq$: Alſo $CPq - PFq = CFq$: And $CPq - PDq = CDq$: But $\overline{CPq - PDq} - \overline{CPq - PFq} = PFq - PDq$; that is, $CDq - CFq = DFq$; and therefore DF is ⊥ CF; Conſequently, (by 6th *Def.* 11 Book *Eucl. El.* and by the 2d of this *Chap.*) ∠ PFD = ∠ PBA: Wherefore in the * Plane ∟d △ PDF (by *Ax.* 1. *Plain Trig.*) PF ‥ PD :: S, ∠ FDP ‥ S, ∠ PFD; that is, S, BP ‥ S, PA :: S, ∠ BAP (*r*) ‥ S, ∠ PBA, or S, B. *Q. E. D.*

* *By* 2. 11. Eucl. El.

AXIOM II.

or $CP_q - PD_q - CP_q + PF_q$

AXIOM II.

In any ∟d *Spheric Triangle*, Radius is to the Sine of the Baſe, as the Tangent of the Angle oppoſite to the Perpendicular, is to the Tangent of the Perpendicular.

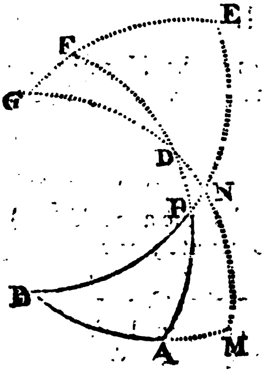
[*See the foregoing Fig.*]

Demonſtration.

By Axiom I. *Pl. Trig.* } FD .. DP : : *r* .. T, ∠ PFD = T, B
} CD .. DP : : *r* .. T, AP.
Wherefore FD × T, B (= DP × *r*) = CD × T, AP;
Conſequently CD .. FD : : T, B .. T, AP.
The △*s* CDF and CAE are Similar; wherefore (by 4. 6. *Eucl. El.*) CD .. FD : : *r* (CA) .. S, BA (AE.)

From the two laſt Analogies you have *r* .. S, BA : : * T, B .. T, PA. *Q. E. D.*

* T, B, or T, ∠ PBA.



For the Solution of the following *Caſes*, I ſuppoſe BPA a ∟d *Spherical Triangle*, and its Sides produced to Quadrants BN, BM, AD: Suppoſe alſo DM, PF, PE, EG, NG, each equal to a Quadrant; then is NE = BP, and the Complement of BA = AM = ∠ ADM, FE = ∠ FPE = ∠ BPA, and GD = NM = ∠ B; and the Angles at A, M, N, E, F, right ones.

N. B. In the following Solutions, △ ſignifies a ∟d *Spheric Triangle.*

The

The Proportions for the Solution of the Sixteen Caſes of Right-angled Spheric Triangles, *with the Solution of their Ambiguities.*

Given.	*Required.*		*Number.*
PA, PB	B	S, BP ·· *r* (S, BN) :: S, PA ·· S, B (S, MN) by *Axiom* I.	1
PA, B	BP	S, B ·· *r* :: S, PA ·· S, BP (*Axiom* 1.) Ambiguous.	2
BP, P	BA	*r* ·· S, BP :: S, P ·· S, BA (*Axiom* 1.)	3
BP,PA	BA	In△DPN ·· S,DP ·· *r*(S,DA) :: S,NP ·· S,D(AM) That is, Σ PA ·· R :: Σ PB ·· Σ, BA.	4
BA,PA	BP	In△PDN.*r*(S,DA) ·· S,D(S,AM) :: S,DP ·· S,PN, That is R ·· Σ BA :: Σ PA ·· Σ PB.	5
BA, P	B	In △PDN.*r* (S,PF) ·· S,PD :: S,P(S,FE) : S,DN, That is, R ·· Σ PA :: S,P ·· Σ B.	6
PA, B	P	In △PDN.S,PD ·· *r* (S,PF) :: S,DN ·· S,P(S,FE) *i. e.* Σ PA ·· R :: Σ B ·· S,P. Ambiguous.	7
B, P	PA	In△PDN.S,P (S,FE) ·· *r*(S,EP) :: S,DN ·· S,DP, *i. e.* S,P ·· R :: Σ B ·· Σ PA.	8
BA,PA	B	S,BA ·· *r*(S,BM) :: T,PA ·· T,B (T,MN) by *Ax.* 2.	9
BA, B	PA	*r* ·· S, BA :: T, B ·· T, PA. (*Axiom* 2.)	10
PA, B	BA	T, B ·· *r* :: T, PA ·· S, BA. Ambiguous.	11
BP, P	PA	In △DGF.*r*(S,F) ·· S,GF :: T,G (T,EN) ·· T,FD, *i. e.* R ·· Σ P :: T, BP ·· T, PA.	12
PA,BP	P	In △ DGF ·· ·· T, G ·· *r* :: T, FD ·· S, FG, *i. e.* T, BP ·· R :: T, PA ·· Σ P.	13
PA, P	BP	In △ DGF ·· S, FG ·· *r* :: T, DF ·· T, G, *i. e.* Σ P ·· R :: T, PA ·· T, BP.	14
BP, P	B	In △ DPN ·· *r* ·· S, PN :: T, P ·· T, DN, That is R ·· Σ BP :: T, P ·· τ, B.	15
B, P	BP	In △ DPN ·· ·· T, P ·· *r* :: T, DN ·· S, PN, *id eſt*, T, P ·· R :: τ B ·· Σ BP.	16

Becauſe *Spheric Trigonometry* is very uſeful, and withal difficult, I thought it not amiſs to add another way of demonſtrating the foregoing Solutions, which is as follows.

Suppoſe in the ∟^{d} *Spheric Triangle* BPA, the Sine of the Hypothenuſe BP = *b*, the Sine of the Perpendicular PA = *p*, and Radius = *r*: Then all the S's, Σ's, T's, &c. of all its Sides and Angles, are juſt Determin'd (or *Given*) and may be expreſs'd in thoſe Symbols without the help of the 2d *Axiom*, but then the Sine of the Baſe BA muſt be firſt deſign'd in the ſaid Symbols, which is done thus.

The △s CDF and CAE in the ſolid *Fig.* in *Page* 345, are Similar; Wherefore (by 4. 6. *Eucl. El.*) CD ·· DF :: CA ·· AE.

That is, $\overline{r^2 - p^2}|^{\frac{1}{2}} \cdot\cdot \overline{b^2 - p^2}|^{\frac{1}{2}} :: r \cdot\cdot \overline{\frac{r^2 b^2 - r^2 p^2}{r^2 - p^2}}\Big|^{\frac{1}{2}} =$ S, AB.

Now by *Axiom* 1.

$$b \cdot\cdot p :: r \cdot\cdot \frac{rp}{b} = \text{S, B}:$$

And * $b \cdot\cdot \overline{\frac{r^2 b^2 - r^2 p^2}{r^2 - p^2}}\Big|^{\frac{1}{2}} :: r \cdot\cdot \overline{\frac{r^4 b^2 - r^4 p^2}{r^2 b^2 - p^2 b^2}}\Big|^{\frac{1}{2}} =$ S, P.

And, ſince by what has been ſaid in *Chap.* 1.

$\overline{Rq - Sq}|^{\frac{1}{2}} = \Sigma$; alſo $\Sigma \cdot\cdot R :: S \cdot\cdot T$; and $S \cdot\cdot \Sigma :: R : \tau$ of any Arc or Angle:

Therefore, $\overline{rr - bb}|^{\frac{1}{2}} = \Sigma$, BP:

And $\overline{r^2 - b^2}|^{\frac{1}{2}} \cdot\cdot r :: b : \frac{rb}{\overline{rr - bb}|^{\frac{1}{2}}} =$ T, BP,

Alſo $\overline{r^2 - p^2}|^{\frac{1}{2}} = \Sigma$, PA:

And $\overline{rr - pp}|^{\frac{1}{2}} \cdot\cdot p :: r : \frac{rp}{\overline{rr - pp}|^{\frac{1}{2}}} =$ T, PA:

Alſo $\overline{r^2 - \frac{r^2 b^2 - r^2 p^2}{r^2 - p^2}}\Big|^{\frac{1}{2}} = \overline{\frac{r^4 - r^2 b^2}{r^2 - p^2}}\Big|^{\frac{1}{2}} = \Sigma$, BA.

---

* *By the* ∟ *the* Hypothenuſe *is limited, but either of the other Sides may be the Perpendicular.*

Alſo $\overline{r^2 - \frac{r^4b^2 - r^4p^2}{r^2b^2 - p^2b^2}}\Big|^{\frac{1}{2}} = \overline{\frac{r^4p^2 - r^2p^2b^2}{r^2b^2 - p^2b^2}}\Big|^{\frac{1}{2}} = \Sigma, P.$

And $\overline{\frac{r^4p^2 - r^2p^2b^2}{r^2b^2 - p^2b^2}}\Big|^{\frac{1}{2}} : r :: \overline{\frac{r^4b^2 - r^4p^2}{r^2b^2 - p^2b^2}}\Big|^{\frac{1}{2}} :$

$$\overline{\frac{r^4b^2 - r^4p^2}{r^2p^2 - p^2b^2}}\Big|^{\frac{1}{2}} = T, P.$$

Alſo $\overline{r^2 - \frac{r^2p^2}{b^2}}\Big|^{\frac{1}{2}} = \overline{\frac{r^2b^2 - r^2p^2}{bb}}\Big|^{\frac{1}{2}} = \Sigma, B.$

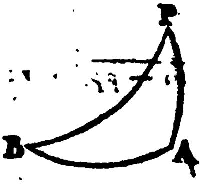
And $\overline{\frac{r^2b^2 - r^2p^2}{b^2}}\Big|^{\frac{1}{2}} : r :: \frac{rp}{b} : \frac{rp}{\overline{b^2 - p^2}|^{\frac{1}{2}}} = T, B.$

And laſtly, $\frac{rp}{b} : \frac{\overline{r^2b^2 - r^2p^2}|^{\frac{1}{2}}}{b} :: r : \frac{\overline{r^2b^2 - r^2p^2}|^{\frac{1}{2}}}{p} = \tau, B.$

The Proportions for Solving the Sixteen Caſes of Right-angled *Spherical Triangles*, with their Ambiguities.

N. B. *The Baſe of any* ∟d *Triangle may be made the Perpendicular, or the Perpendicular the Baſe, wherefore both are called Legs.*

Given the Leg PA, Hypothenuſe PB: Required Angle oppoſite PA, viz. ∠ B.



Solution.

S, BP : r :: S, PA : S, B.

Demonſtration.

Their reſpective Values are Proportional; that is,

$$b : r :: p : \frac{pr}{b}.$$ Q. E. D.

CASE II.

Given Leg PA, Angle oppoſite, viz. B: Required Hypothenuſe BP.

Solution and *Demonſtration.*

It is evident from and by the firſt *Caſe*, (by only inverting the Proportion) that

S, B ·· r :: S, PA ·· S, BP. Q. E. D.

Note; *That this Caſe is Ambiguous, that is,* B P *may be* ⊏ *or* ⊐ *Quadrant.*

C A S E III.

Given Hypothenuſe BP, Angle P; Required Side oppoſite ∠ P, towit BA.

Solution.

r ·· S, BP :: S, P ·· S, BA.

Demonſtration.

$$r \cdot\cdot b :: \overline{\frac{r^4 b^2 - r^4 p^2}{r^2 b^2 - b^2 p^2}}\Big|^{\frac{1}{2}} \cdot\cdot \overline{\frac{r^2 b^2 - r^2 p^2}{r^2 - p^2}}\Big|^{\frac{1}{2}}$$ Q. E. D.

C A S E IV.

Given Leg PA, Hypothenuſe PB: Required Leg BA.

Solution.

Σ PA ·· r :: Σ BP ·· Σ BA.

Demonſtration.

$$\overline{rr - pp}\big|^{\frac{1}{2}} \cdot\cdot r :: \overline{rr - bb}\big|^{\frac{1}{2}} \cdot\cdot \overline{\frac{r^4 - r^2 b^2}{r^2 - p^2}}\Big|^{\frac{1}{2}}$$ Q. E. D.

C A S E V.

Given the Legs BA and PA Severally: Required the Hpyothenuſe BP.

Solution and *Demonſtration.*

It is evident by *Caſe* IV. That

r ·· Σ BA :: Σ PA ·· Σ BP. Q. E. D.

C A S E VI.

Given Leg PA, ∠ adjoyning P: Required ∠ B.

Solution.

$r \cdot\cdot \Sigma PA :: S, P :: \Sigma, B$

Demonſtration.

$$r \cdot\cdot \left.\overline{rr - pp}\right|^{\frac{1}{2}} :: \left.\overline{\frac{r^4 b^2 - r^4 p^2}{r^2 b^2 - b^2 p^2}}\right|^{\frac{1}{2}} \cdot\cdot \left.\overline{\frac{r^2 b^2 - r^2 p^2}{b^2}}\right|^{\frac{1}{2}} \text{ Q.E.D.}$$

CASE VII.

Given Leg PA, and ∠ oppoſite, *viz.* B : Required ∠ P.

Solution and *Demonſtration.*

It is evident by the VIth *Caſe*, that

$\Sigma PA \cdot\cdot r :: \Sigma B \cdot\cdot S, P$. Ambiguous. *Q. E. D.*

CASE VIII.

Given the ∠s B and P ſeverally: Requir'd Leg PA.

Solution and *Demonſtration.*

It is evident by the VIth *Caſe*, that

$S, P \cdot\cdot r :: \Sigma B \cdot\cdot \Sigma PA$. *Q. E. D.*

CASE IX.

Given the Legs BA and PA ſeverally: Required ∠ B.

Solution.

$S, BA \cdot\cdot r :: T, PA \cdot\cdot T, B.$

Demonſtration.

$$\left.\overline{\frac{r^2 b^2 - r^2 p^2}{r^2 - p^2}}\right|^{\frac{1}{2}} \cdot\cdot r :: \frac{pr}{\left.\overline{r^2 - p^2}\right|^{\frac{1}{2}}} \cdot\cdot \frac{rp}{\left.\overline{b^2 - p^2}\right|^{\frac{1}{2}}} \text{ Q. E. D.}$$

CASE X.

Given Leg BA, ∠ adjacent, *viz.* B : Required Leg PA.

Solution and *Demonſtration.*

It is evident by the 9th *Caſe*, that

$r \cdot\cdot S, BA :: T, B \cdot\cdot T, PA$. *Q. E. D.*

CASE XI.

Given Leg PA, ∠ opposite, to wit B; Required Leg BA.

Solution and *Demonſtration.*

It is evident by the 9th *Caſe*, that

T, B ·· r :: T, PA ·· S, BA. Ambiguous. *Q. E. D.*

CASE XII.

Given the Hypothenuſe, BP, ∠ P: Required Leg PA.

Solution.

r ·· Σ P :: T, BP ·· T, PA.

Demonſtration.

$$r \cdot\cdot \overline{\frac{r^4p^2 - r^2b^2p^2}{r^2b^2 - b^2p^2}}\Big|^{\frac{1}{2}} :: \frac{rb}{\overline{r^2 - b^2}|^{\frac{1}{2}}} \cdot\cdot \frac{pr}{\overline{r^2 - p^2}|^{\frac{1}{2}}}$$ Q. E. D.

CASE XIII.

Given Leg PA, Hypothenuſe BP: Requir'd ∠ adjacent, *viz.* P.

Solution and *Demonſtration.*

It is evident by the 12th *Caſe*, that

T, BP ·· r :: T, PA ·· Σ P. *Q. E. D.*

CASE XIV.

Given Leg PA, ∠ P; Required Hypothenuſe BP.

Solution and *Demonſtration,*

It is evident by the 12th *Caſe*, that

Σ P ·· r :: T, PA ·· T, BP. *Q. E D.*

CASE XV.

Given Hypothenuſe BP, ∠ P: Required ∠ B.

Solution.

r ·· Σ, BP :: T, P ·· τ B.

Demon-

Demonſtration.

$$r \,\because\, \overline{rr - bb}\,\Big|^{\frac{1}{2}} :: \overline{\frac{r^4h^2 - r^4p^2}{r^2p^2 - h^2p^2}}\,\Bigg|^{\frac{1}{2}} \because \overline{\frac{r^2h^2 - r^2p^2}{p^2}}\,\Bigg|^{\frac{1}{2}}$$ Q. E. D.

Note, This 15th *Caſe* is generally ſolv'd thus, τ, P ·· r : : Σ, BP ·· τ, B.

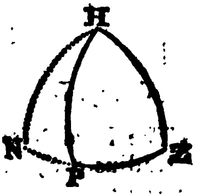
So many of the other *Caſes* may be ſolv'd otherwiſe than as they are ſolv'd.

CASE XVI.

Given the ∠s B and P ſeverally: Required to find the *Hypothenuſe* BP.

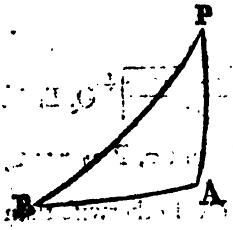
Solution and Demonſtration.

It is evident by the 15th *Caſe*, that T, P ·· r : : r, B ·· s BP. *Q. E. D.*



Any oblique Quadrantal Triangle ZHP, whoſe Side HZ is a Quadrant, may be reduc'd to the ∟d Triangle HNP, by producing the other ſide ZP to N, ſo that ZN be a Quadrant; (or taking away a Quadrant, if ZP had been bigger than a Quadrant) Then in the ∟d *Spheric Triangle* HNP, NH is = ∠ Z, and PN the Complement of ZP, and ∠ PHN the Complement of ZHP.

The Ingenious Lord *Neper*, from a diligent Obſervation of the Proportions for ſolving the foregoing 16 *Caſes*, found they may be ſolv'd by one general Propoſition; for the underſtanding of which 'tis neceſſary to premiſe, that he calls the two Legs (*viz.* the Sides including the Right-angle) together with the Complements of the other three Parts (*viz.* of the *Hypothenuſe* and two Oblique Angles) five Circular Parts, taking no Notice of the Right-angle as being known. And if the three Parts which enter the Queſtion (of which two are given and one requir'd) have no Interruption (a Right-angle coming between being accounted none) that Part which is between the other two is called the *middle Part*, and the other two, *Extreams Conjunct*: But if there be an Interruption, that Part which is ſeparated from the other two is called the *middle Part*, and the other two, *Extreams Oppoſite or Disjunct*.



For Inſtance, In the ∟d Triangle BPA ;

		Extr. Conj.		Extr. Disj.
BA	be middle Part, then	Comp. B PA	And	Comp. P Comp. BP
PA		BA Comp. P		Comp. BP Comp. B
Comp. P		Comp. BP PA		Comp. B BA
Comp. BC		Comp. B. Comp. P.		BA PA
Comp. B		BA Comp. BP		Comp. P PA

The Lord *Neper*'s general Propoſition is thus,

Radius × S, middle Part = ☐ Tangents of the *Extreams Conjunct.*

And R × S, middle Part = ☐ Co-ſines of the *Extreams Disjunct.*

That is, The Radius multiply'd by the Sine of the middle Part is equal to the Rectangle, or Product of the Tangents of the Extreams Conjunct, *and = to the Rectangle of the Co-ſines of the* Extreams Disjunct.

Hence to find the middle Part when the other Parts given are either *Extreams* $\left\{\begin{array}{l}\textit{Conjunct}\\ \textit{Disjunct.}\end{array}\right\}$

RULE I.

$\left\{\begin{array}{l}\text{R .. T, one Extr. :: T, other Extr. .. S, middle Part.}\\ \text{R .. } \Sigma\text{, one Extr. :: } \Sigma\text{, other Extr. .. S, middle Part.}\end{array}\right.$

Here you may obſerve, that when the middle Part is to be found, you muſt begin with the *Radius*: But when you are to find an Extream either $\left\{\begin{array}{l}\textit{Conjunct}\\ \textit{Disjunct.}\end{array}\right\}$

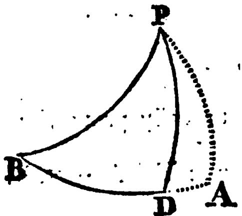
RULE II.

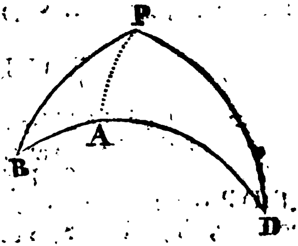
RULE II.

{T, one Extr. ·· R :: S, middle Part ·· T, other Extr.
{Σ, one Extr. R :: S, middle Part ·· Σ, other Extr.

So that when one Extream is to be found, the other Extream muſt be the firſt Term.

Of the Solution of Oblique Spheric Triangles.





Oblique *Spheric Triangles* may be reduc'd to two ∟d *Spheric Triangles*, by letting fall a Perpendicular; which Perpendicular either divides the Oblique propos'd into two Right, or makes two Right by adding a ∟d Triangle to it.

In Oblique Triangles there are Twelve *Caſes*, Ten of which may be ſolv'd (being thus prepar'd) by the two firſt *Axioms*, or by Rules deduc'd from them.

RULE I.

The Co-ſines of the Angles at the Baſe are proportional to the Sines of the Angles at the Vertex.

Note, That BA and DA are the Baſes of the ∟d Triangles BPA, DPA; the Angles B and D are the Angles at the Baſe, and the ∠s BPA DPA are called the Angles at the Vertex; and BP and DP are called the Sides.

For by the ſixth *Caſe* {*r* ·· Σ, PA :: S, BPA ·· Σ, B.
{*r* ·· Σ, PA :: S, DPA ·· Σ, D.

Therefore, Σ, B ·· S, BPA :: Σ, D ·· S, DPA.

RULE II.

The Co-ſines of the Sides are proportional to the Co-ſines of the Baſes:

For by the 5th *Caſe* $r \cdot\cdot \Sigma, PA :: \left\{\begin{array}{l}\Sigma, BA \cdot\cdot \Sigma, BP. \\ \Sigma, DA \cdot\cdot \Sigma, DP.\end{array}\right.$

Therefore, $\Sigma, BA \cdot\cdot \Sigma, BP :: \Sigma, DA \cdot\cdot \Sigma, DP.$

RULE III.

The Sines of the Baſes are reciprocally proportional to the Tangents of the Angles at the Baſe:

For by the 2d *Axiom* $\left\{\begin{array}{l}S, BA \cdot\cdot r :: T, PA \cdot\cdot TB \\ S, DA \cdot\cdot r :: T, PA \cdot\cdot T, D.\end{array}\right.$

Therefore, $S, BA \times T, B \; (= r \times T, PA) = S, DA \times T, D,$

Conſequently, $S, BA \cdot\cdot S, DA :: T, D \cdot\cdot T, B.$

RULE IV.

The Tangents of the Sides are reciprocally proportional to the Co-ſines of the Angles at the Vertex.

For, by the 12th *Caſe* $T, BP \cdot\cdot r :: T, PA \cdot\cdot \Sigma, BPA:$ And $T, DP \cdot\cdot r :: T, PA \cdot\cdot \Sigma, DPA.$

Therefore, $T, BP \times \Sigma, BPA \; (= r \times T, PA) = T, DP \times \Sigma, DPA.$

Conſequently, $\left\{\begin{array}{l}T, BP \cdot\cdot T, DP :: \Sigma, DPA \\ \cdot\cdot \Sigma, BPA.\end{array}\right.$

AXIOM III.

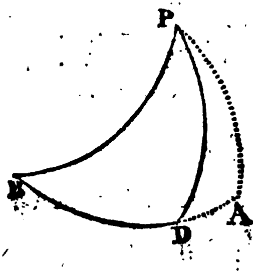
In any Triangle (whether Right-angled or Oblique) the Sines of the Sides are proportional to the Sines of the oppoſite Angles.

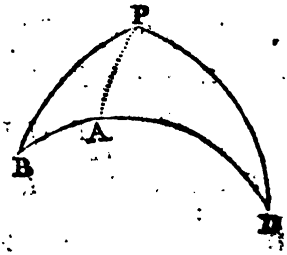
Demonſtration.

By the firſt *Axiom* $\left\{\begin{array}{l}S, BP \cdot\cdot r :: S, PA \cdot\cdot S, B \\ S, DP \cdot\cdot r :: S, PA \cdot\cdot S, D.\end{array}\right.$

Therefore, $S, BP \times S, B \; (= r \times S, PA) = S, DP \times S, D;$

Wherefore, $S, BP \cdot\cdot S, DP :: S, D \cdot\cdot S, B.$ *Et ſic de cæteris Lateribus & angulis.* Q. E. D.





Obſerve this Rule *for letting fall your Perpendicular.*

Let it fall from the end of a given Side, and oppoſite to a given Angle; for, by ſo doing, you have enough given in one of the ∟d *Spheric Triangles* to determine any of its unknown Parts.

The two firſt *Caſes* are ſolved each by one Operation by the third *Axiom* foregoing.

The next Eight *Caſes* are ſolved each by two O[illegible]ions. And that you may know what Part of the ∟d *Spheric Triangle* to find in your firſt Operation; Obſerve to find ſuch a part, which, when found, the other part of the thing required (which is to be added to, or ſubtracted from the firſt Part found) may come under one of the four *Rules* mention'd.

This will be eaſily underſtood from the general View of the Proportion for the Solution of the firſt Ten *Caſes* of Oblique Triangles annex'd.

Proportions for the ſolving the firſt Ten Caſes of Oblique Spheric Triangles.

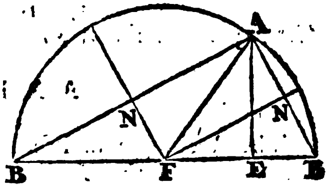
[*See the foregoing Fig.*]

Given.	*Req.*	*Proportions.*	*Caſe.*
B. BP, PD	D	S, PD ·· SB :: S, BP ·· S, D. Ambiguous.	1
BP, B, D	PD	S. D ·· S, BP :: S, B ·· S, PD. Ambiguous.	2
BP, PD, B	BD	*Firſt*, Find BA by the 12th *Caſe*; thus, *r* ·· Σ, B :: T, BP ·· T, BA. And then by the ſecond *Rule* Σ, BP ·· Σ, BA :: Σ, DP ·· Σ, DA: Then BA ± DA = BD, according as the Perpendicular falls within or without the Triangle, which is doubtful, unleſs the kind of Angle D were known.	3
BP, PD, B	P	*Firſt*, Find BPA by the 15th *Caſe*; thus, Σ, BP ·· *r* :: *τ*, B ·· T, BPA. Then by the fourth *Rule* T, DP ·· T, BP :: Σ, BPA ·· Σ, DPA. Then BPA ± DPA = PBD. And here the following of PA within or without is doubtful, unleſs the kind of Angle D be known.	4

Given.	*Req.*	*Proportions.*	*Caſe.*
BP, B, D	P	*Firſt*, Find BPA by the 15th Caſe; thus, Σ, BP ·· r :: τ, B ·· T, BPA. Then by the firſt *Rule* Σ, B ·· S, BPA :: Σ, D ·· S, DPA. Then BPA ± DPA = BPD: That is, if B and D are $\left\{\begin{matrix}\text{alike}\\ \text{unlike.}\end{matrix}\right.$	5
BP, B, D	BD	*Firſt*, Find BA by the 12th Caſe; thus, Σ, B ·· r :: τ BP ·· τ BA. Then by the third *Rule* T, D ·· T, B :: S, BA ·· S, DA. Then BA ± DA = BD: If B and D are $\left\{\begin{matrix}\text{alike}\\ \text{unlike.}\end{matrix}\right.$	6
BP, B, P	D	*Firſt*, Find BPA by the 15th Caſe; thus, Σ, BP ·· r :: τ, B ·· T, BPA. Then by the firſt *Rule* S, BPA ·· S, DPA :: Σ B ·· Σ, D. If BPA is $\sqsubset$ BPD, and alſo B $\left\{\begin{matrix}\text{Acute}\\ \text{Obtuſe,}\end{matrix}\right.$ D is $\left\{\begin{matrix}\text{Obtuſe}\\ \text{Acute:}\end{matrix}\right\}$ But if BPA is $\sqsupset$, BPD and B $\left\{\begin{matrix}\text{Acute}\\ \text{Obtuſe,}\end{matrix}\right\}$ D is $\left\{\begin{matrix}\text{Acute}\\ \text{Obtuſe.}\end{matrix}\right.$	7
B, P, BP	PD	*Firſt*, Find BPA by the 15th Caſe; thus, Σ, BP ·· r :: τ B ·· T, BPA. Then by the fourth *Rule* Σ, DPA ·· Σ, BPA :: T, BP ·· T, DP. If DPA is $\left\{\begin{matrix}\text{like}\\ \text{unlike,}\end{matrix}\right\}$ B (or PA) DP is $\left\{\begin{matrix}\text{leſs}\\ \text{greater}\end{matrix}\right\}$ than a Quadrant.	8
BP, BD, B	DP	*Firſt*, Find BA by the 12th Caſe; thus, Σ, B ·· r :: τ BP ·· τ BA. Then by the ſecond *Rule* Σ, BA ·· Σ, BP :: Σ, DA ·· Σ, DP. If DA is $\left\{\begin{matrix}\text{like}\\ \text{unlike}\end{matrix}\right\}$ (PA) ∠ B, then BD is $\left\{\begin{matrix}\sqsupset\\ \sqsubset\end{matrix}\right\}$ than a Quadrant.	9
BP, BD, B	D	*Firſt*, Find BA by the 12th Caſe; thus, Σ, B ·· r :: τ, BP ·· τ, BA. Then by the third *Rule* S, DA ·· S, BA :: T, B ·· T, D. If BA is $\left\{\begin{matrix}\sqsupset\\ \sqsubset\end{matrix}\right\}$ BD, then D is $\left\{\begin{matrix}\text{like}\\ \text{unlike}\end{matrix}\right\}$ B.	10

LEMMA I.

LEMMA I.

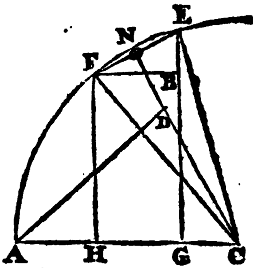


$\frac{1}{2} r \times$ U is = Sq : $\frac{1}{2}$ Arch.

Demonſtration.

The $\llcorner^{d}$ $\triangle s$ FNB and AEB have the $\angle$ B common: Wherefore (by 4. 6. *Eucl. El.*) EB ·· BN :: AB ·· FB :: $\frac{1}{2}$ AB (BN) ·· $\frac{1}{2}$ FB: Wherefore $\frac{1}{2}$ FB × EB = BNq: *i. e.* $\frac{1}{2} r \times$ U is = Sq : $\frac{1}{2}$ Arch. *Q. E. D.*

LEMMA II.



The Difference of the Vers'd Sines of two Arcs multiply'd by half Radius, is equal to the Sine of half the Sum of thoſe Arcs multiply'd by the Sine of half the Difference of the Arcs.

Suppoſe the two Arcs to be AE the Greater, and AF the Leſſer, then their Difference is FE. Now take FN = NE, and then NE will be half their Difference: And conſequently, NA is half their Sum. Draw the Right Lines AC, EC, FC, NoC; and draw EG and FH $\perp^{s}$ AC, AD $\perp$ NC; as alſo FB ∥ AC, and draw the Right Line FoE: Then HG = FB is the Difference of the Vers'd Sines of the aforeſaid Arcs. AD is the Sine of half their Sum, and $\frac{1}{2}$ FE the Sine of half their Difference.

Now, what I am to prove is, that HG (FB) × $\frac{1}{2}r$ is = AD × $\frac{1}{2}$ FE.

Demonſtration.

Fo and AD being each perpendicular to NC, are therefore parallel: And FB is (by Suppoſition) parallel AC; conſequently, the $\angle$ oFB, or EFB is = $\angle$ DAC, and the $\angle$ FBE being alſo = $\llcorner$ = $\angle$ ADC, the $\triangle s$ EFB and CAD are ſimilar; conſequently, (by 4. 6. *Eucl. El.*)

FB .. FE :: AD .. AC:

Therefore, FB .. $\frac{1}{2}$ FE :: AD .. $\frac{1}{2}$ AC: That is, becauſe AC $=$ r, FB .. $\frac{1}{2}$ FE :: AD .. $\frac{1}{2}$ r.

Conſequently, FB $\times \frac{1}{2} r = \frac{1}{2}$ FE $\times$ AD. *Q. E. D.*

AXIOM IV.

The Rectangle or Product of the Sines of the Legs, is to the Square of the *Radius*, as the Difference of the Vers'd Sines of the Baſe, and of the Difference of the Legs, is to the Vers'd Sine of the Vertical Angle.

That is, the Rectangle of the Sines of the Legs, is to the Square of the Radius, as the Difference between the versed Sine of the Base and the versed Sine of the Difference of the Legs, is to the versed Sine of the vertical Angle.

Suppose Pβ perpendicular to the Plane CBα, and Pb perpendicular to CA.

Suppoſe B to be the Angle required of the *Oblique Spheric Triangle* BPA, fitted as in the annex'd Figure, where PF ($=$ FM) and AE are the Sines of the Legs BP, ($=$ BM) and BA reſpectively; and P*b* the Sine of the Baſe PA; AL the Vers'd Sine of MA the Difference of the Legs, and *b*L $=$ *b*A $-$ LA, the U of Baſe $-$ U of the Difference of the Legs. Imagine the $\perp$*s* P*β* and *wy* dropt from P and *w* on the Plane BAM*α*C; and draw C*w* and C*α* $\perp$*s* BC, and draw the Line *βb*.

Now, what I am to prove is, that

PF ($=$ MF) $\times$ AE .. rr (AC $\times$ *α*C) :: *b*L .. *yα*.

Demon.

b β   47.

1.   r r

−   b q

=   ere-

fore   β F

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(the

T   ere-

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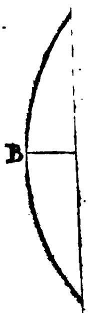
6.



he

nd

= Bas, and

cr. to denote the Legs. Then,

Solution.

$$S,m \times S,n :: rr :: S\overline{\tfrac{1}{2}B + \tfrac{1}{2}\text{Diff. } cr.} \times S\overline{\tfrac{1}{2}B - \tfrac{1}{2}\text{Diff. } cr.} :: Sq : \tfrac{1}{2}\text{ Angle.}$$

[*See the foregoing Solid Fig.*]

Suppoſe B to be the Angle required of the [illegible] *Triangle* BPA, fitted as in the annex'd Figure, where PF (= FM) and AE are the Sines of the Legs BP, (= BM) and BA reſpectively; and P*b* the Sine of the Baſe PA; AL the Vers'd Sine of MA the Difference of the Legs, and *b*L = *b*A − LA, the U of Baſe − U of the Difference of the Legs. Imagine the ⊥s P*β* and *wy* dropt from P and *w* on the Plane BAM*α*C; and draw C*w* and C*α* ⊥s BC, and draw the Line *βb*.

Now, what I am to prove is; that

$$PF(=MF)\times AE : rr\,(AC\times\alpha C) :: bL : y\alpha.$$

Demon-

Demonſtration.

$b\beta$ is $\perp P\beta$ (by 3. *Def.* 11 Book *Eucl. El.*) Wherefore (by 47. 1. *Eucl. El.*) $Pbq: - P\beta q: = b\beta q:$; Alſo $bCq: = rr - Pbq:$; and $\beta Cq = rr - P\beta q:$; But $\overline{rr - P\beta q}: - \overline{rr - Pbq} = Pbq: - P\beta q$; that is, $\beta Cq: - bCq: = b\beta q$: Therefore, βb is $\perp bC$. In like manner you may prove that βF is $\perp CF$; and therefore $P\beta$ falls on the Line BM; and wy (the S, ∠ B) falls on Ca.

The $\triangle^s$ AEC, (DFC) DLM, $Db\beta$ are ſimilar; therefore (by 4. 6. *Eucl. El.*) $AE \cdot\cdot AC :: (DL \cdot\cdot DM)\ bL \cdot\cdot \beta M$.

The $\triangle^s$ $PF\beta$ and wCy having the $\angle^s$ at β and y each = ∟, and the $\angle^s$ $PF\beta$ and wCy equal, each being (by *Def.* 6. 11. *Eucl. El.*) = ∠ B, are ſimilar: Wherefore (by 4. 6. *Eucl. El.*)

$$PF \cdot\cdot F\beta :: wC \cdot\cdot Cy:$$

$$i.e.\ MF \cdot\cdot F\beta :: aC \cdot\cdot Cy:$$

$$\text{Therefore, } MF \cdot\cdot MF - F\beta :: aC \cdot\cdot aC - Cy;$$

$$i.e.\ MF \cdot\cdot \beta M :: aC \cdot\cdot ya:$$

$$\text{Conſequently, } MF \cdot\cdot aC :: \beta M \cdot\cdot ya.$$

And multiplying the correſpondent Terms of this and the foregoing Proportion, you'll have $AE \times MF \cdot\cdot AC \times aC :: (bL \times \beta M \cdot\cdot \beta M \times ya)\ bL \cdot\cdot ya$. Q. E. D.

CASE XI.

The three Sides of any *Spheric Triangle* being given to find an Angle:

Suppoſe m and n to be = the two Legs, B = Baſe, and $cr.$ to denote the Legs. Then,

Solution.

$$S,m \times S,n \cdot\cdot rr :: S\,\overline{\tfrac{1}{2}B + \tfrac{1}{2}\text{Diff. } cr.} \times S\,\overline{\tfrac{1}{2}B - \tfrac{1}{2}\text{Diff. } cr.} \cdot\cdot Sq: \tfrac{1}{2}\text{ Angle.}$$

[*See the foregoing Solid Fig.*]

Demonſtration.

$$\mathrm{AE}\times\mathrm{MF} \,.\, rr :: \begin{cases} bL \,.\, ya \text{ (by } \textit{Axiom } 4.) \\ bL \times \tfrac{1}{2}r \,.\, ya \times \tfrac{1}{2}r. \\ \overline{S\,\tfrac{1}{2}B + \tfrac{1}{2}\,\text{Diff. cr}} \times \overline{S\,\tfrac{1}{2}B - \tfrac{1}{2}\,\text{Diff. cr}} \\ :: \text{Sq. } \tfrac{1}{2} \text{ Angle (by } \textit{Lemmas } 1,\, 2.) \\ \text{Q. E. D.} \end{cases}$$

CASE XII.

The three Angles being given to find a Side.

Solution.

The Angles adjacent to the Side required call Legs, and the Angle oppoſite call the Baſe; then work as in the 11th Caſe.

For ſuch is the Operation in the ſupplemental Triangle, whoſe Angles and Sides are equal to the Supplements of the Sides and Angles of the Triangle propos'd: And Arcs and their Supplements have the ſame Sines and Tangents.

PART II.

PART II.

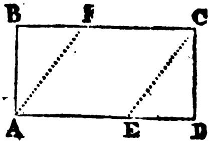
The Solution of ſome Problems in Plane Geometry.

CHAP. I.

Of Problems producing Simple Equations.

Problem I.

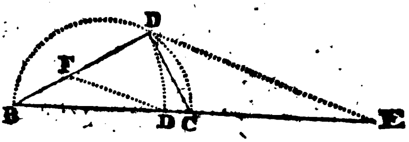
To inſcribe a *Rhombus* in a given Oblong, *i. e.* having the Sides of the $\llcorner^{d}$ ▭; *viz.* AB and BC given: To find the Segment BF or DE, which being cut off, the Remainder FC or AE will be the Side of the *Rhombus* ſought.



Solution.

Let	1	$AB = b\ (= DC) = 20.$
	2	$BC = C\ (= AD) = 40.$
Suppoſe	3	$BF = x\ (= DE)$
Then	4	$FC = c - x\ (= BC - BF) = FA.$
47. 1. *Eucl. El.*	5	$bb + xx = FAq = cc - 2cx + xx.$
$5 - bb - xx + 2cx$	6	$+ 2cx = cc - bb.$
$6 \div 2c$	7	$x = \frac{cc - bb}{2c} = 15.$

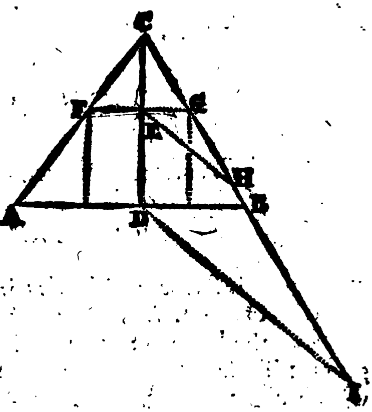
Conſtruction.



Having deſcrib'd a Semicircle on BC = c, apply therein CD = b, and draw BD, which being ⊥ CD (by 31. 3. *Eucl. El.*) is (by 47. 1. *Eucl. El.*) = $\overline{cc-bb}|^{\frac{1}{2}}$: Then make as BE (= $2c$) ∴ BD ($\overline{cc-bb}|^{\frac{1}{2}}$) ∷ BD ∴ FB = $\frac{cc-bb}{2c} = x$.

Problem II.

To inſcribe the greateſt Square poſſible in a given △; *i. e.* having given the Height CD of the △ ACB, and the Baſe AB to find a Portion of the Altitude, *viz.* CE, which being cut off, there ſhall remain ED = FG.



Solution.

Suppoſe	1	AB = b = 24.
	2	CD = p = 18.
	3	CE = a = ?
Then	4	ED = $p - a$ = FG.

The Line FFG being drawn ‖ AB, the △s CAB and CFG will be ſimilar; wherefore

By 4. 6. *Eucl. El.*	5	$b \therefore p :: \overline{p-a} \therefore a$.
Wherefore	6	$ba = pp - pa$.
$6 + pa$	7	$ba + pa = pp$.
$7 \div \overline{b+p}$	8	$a = \frac{pp}{b+p} = 7\frac{5}{7}$

Conſtruction.

Upon the Side of the Triangle, *viz.* CB produc'd, make CH $= p$, and HI $= b$; ſo that the whole Line ſhall be $= p + b$, and having joyned ID, draw HE parallel to it, which will cut CD in the Point E required.

For $CI \cdot\cdot CD :: CH \cdot\cdot CE$ by 2. 6. *Eucl. El.* That is,

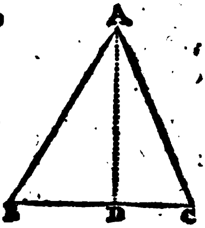
$$b + p \cdot\cdot p :: p \cdot\cdot a.$$

Conſequently, $a = \frac{pp}{b + p}$ *ut ſupra.*

Problem III.

In an acute angled △ having the three Sides given ſeverally to determine the Point where a Line perpendicularly let fall from the Vertex ſhall cut the Baſe; *i. e.*

Having given AB, AC, BC ſeverally to find BD in the annex'd Fig.

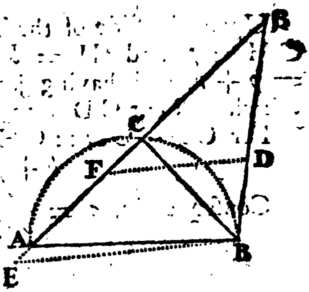


Solution.

Suppoſe it done, and the Line (or ⊥) AD drawn; then

Put	1	$BC = b = 9.$
	2	$CA = c = 8.$
	3	$AB = d = 11.$
And ſuppoſe	4	$BD = a = ?$
Then	5	$DC = b - a.$
By 47. 1. *Eucl. El.*	6	$dd - aa = ADq: = cc - bb + 2ba + aa.$
$6 + aa - cc + b^2$	7	$dd - cc + bb = 2ba.$
$7 \div 2b$	8	$\frac{dd - cc + bb}{2b} = a = 7\frac{2}{3}.$

Construction.

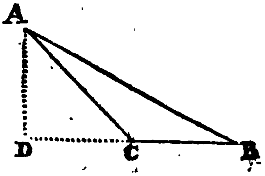


On A B the longeſt Leg of the given △ having deſcrib'd a Semicircle, apply therein AC the other Leg, and draw the Line B C; then will BC be = $\overline{dd - cc}|^{\frac{1}{2}}$; produce A C to β, making Cβ = b; and draw the Line βB; then will βBq be = $\overline{dd - cc + bb}$, which, ſince it is to be divided by $2b$, make as βE = $2b$ ∴ βB = $\overline{dd - cc + bb}|^{\frac{1}{2}}$ ∷ βF = Bβ ∴ βD the Segment ſought, = BD = a.

Problem IV.

In an Obtuſe angled △ having the three Sides given ſeverally to determine the Point where a Line let fall perpendicularly from the Vertex ſhall cut the Baſe produced; *i. e.*

Having given AB, BC, AC ſeverally in the annex'd Fig. to find CD



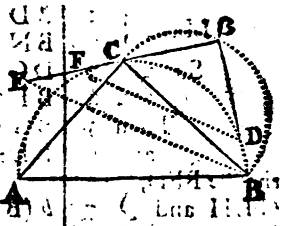
Suppoſe it done, and A D drawn; then

Solution.

Suppoſe	1	BC = b = 10.
	2	AC = c = 16.
	3	AB = d = 24.
	4	DC = a = ?
Then	5	BD = $a + b$
47. 1. *Eucl. El.*	6	$dd - aa - 2ab - bb =$ DAq $= cc - aa.$
$6 + a^2 + 2ab - cc$	7	$dd - bb - cc = 2ab.$
$7 \div 2b$	8	$\frac{dd - bb - cc}{2b} = a = 11.$

Con-

Conſtruction.

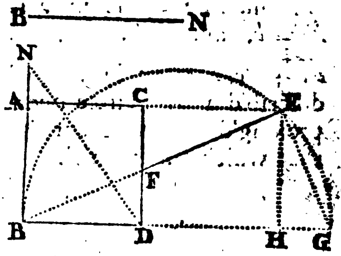


On the longeſt Leg AB of the given Triangle deſcribe a Semicircle, and in it apply AC the other Leg, drawing alſo CB; ſo will CBq: be $= dd - cc$: Then on CB deſcribe another Semicircle, and therein apply Cβ the Baſe of the Triangle given, drawing likewiſe the Line Bβ: Then will βBq be $= dd - cc - bb$: And conſequently $\beta B = \sqrt{dd - cc - bb}$: And ſince the firſt part of the eighth Step is divided by $2b$, make as βE $= 2b$ . . $\beta B = \overline{d^2 - c^2 - b^2}|^{\frac{1}{2}}$ :: $\beta F = \beta B$ . . $\beta D = DC$ the Segment ſought $= a$.

* * *

CHAP. II.

A Problem producing a Simple Quadratick Equation.



HAving given the Square AD, and a Right Line BN, you are to produce the Side AC to E, ſo that EF drawn from E towards B, ſhall be equal to BN.

It will be evident, if you imagine a Semicircle to paſs thro the Points B and E, that the moſt commodious way will be to find the Line DG, that you may have the Diameter BG, upon which having afterwards deſcrib'd a Semicircle, there will be need of no other Operation to ſatisfy the Problem, than to produce the Side AC till it occur the preſcrib'd Periphery.

Solu-

Solution.

Make	1	$BD = b = DC = 3.$
	2	$BN = c = FE = 4.$
Suppoſe	3	$DG = x.$
and	4	$BF = y.$
Then	5	$BG = b + x.$
and	6	$BE = y + c.$
the △s BFD, GEH and BEG are ſimilar	7	$b\,(BD) \cdot\cdot\, y\,(BF) :: b\,(EH) \cdot\cdot\, y\,(EG.)$
	8	$b + x\,(BG) \cdot\cdot\, y\,(EG) :: y + c\,(BE) \cdot\cdot\, b\,(EH.)$
Equation	9	$bb + bx = yy + yc.$
47. 1. *Eucl. El.*	10	$bb + 2bx + xx\,(BGq) = yy\,(EGq)$ $+ yy + 2yc + cc\,(BEq;)$
10 − 9	11	$bx + xx = yy + yc + cc.$
11 − cc	12	$bx + xx - cc = yy + yc.$
12 = 9	13	$bx + xx - cc\,(= yy + yc) = bb + bx.$
$13 - bx + c^2$	14	$xx = bb + cc.$
14 ⱳ 12	15	$x = \sqrt{bb + cc} = 5.$

Conſtruction.

Having produc'd the Side of the Square BA to N, ſo that BN ſhall be = the given Right Line BN; then ſince BD is $= b$, and $BN = c$, the Hypothenuſe DN will be $= \overline{bb + cc}|^{\frac{1}{2}} = x$: Having therefore made DG = DN, and deſcrib'd a Semicircle upon the whole Line BG, if AC be prolong'd until it occur the Periphery in E, you'll have done that which was requir'd.

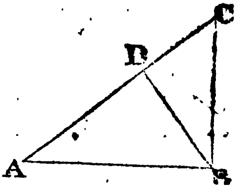
CHAP. III.

CHAP. III.

Of PROBLEMS *producing Adfected Quadratick Æquations.*

PROBLEM I.

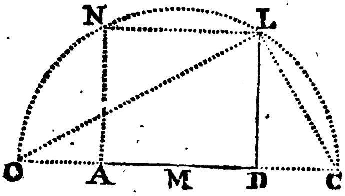
IN the $\llcorner^{d}$ △ ABC, the Perpendicular, BC, and the alternate Segment of the Hypothenuſe, (made by a ⊥ let fall from the ∟) *viz.* AD being given, to find the other Segment DC, *&c.*



Solution.

Let	1	$BC = p =$ the $\perp = 45$
	2	$b = AD$ the alternate Segment $= 48$
And ſuppoſe	3	$a = DC$
By 4. 6. *Eucl. El.*	4	$b \,..\, BD :: BD \,..\, a$
Æq.	5	$ba = BDq :$
By 47. 1. *Eucl. El.*	6	$pp - aa = BDq :$
$5 = 6$	7	$ba = pp - aa$
$7 + aa$	8	$a^2 + ba = pp$. Caſe 1. *Of Adfected Quadratick Æquations.*
Comp. □	9	$a^2 + ba + \frac{1}{4}bb = pp + \frac{1}{4}bb$
$9 \omega 2$	10	$a + \frac{1}{2}b = \sqrt{pp + \frac{1}{4}bb}$
$10 - \frac{1}{2}b$	11	$a := \sqrt{pp + \frac{1}{4}bb} - \frac{1}{2}b = 27.$

Conſtruction.

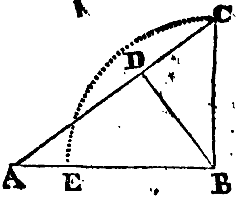


With AD = *b*, and the Perpendicular = *p*, make the $\llcorner_{d}$ ▭ ADLN; then upon M, the middle Point of AD, deſcribe ſuch a Semicircle as will paſs thro' L and N the remoteſt Points of the ▭; then compleat the Diameter, by producing A D to O and C; then will D C, or O A, be $= a$; for

$$OD\ (a+b)\ .\ .\ DL\ (p)\ ::\ DL\ (p)\ .\ .\ DC = a$$

per 13. 6 *Eucl. El.* conſequently $aa + ba = pp$; as in the 8th Step.

PROBLEM II.

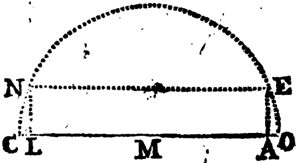


The Difference A E between the Baſe A B and Perpendicular B C of the $\llcorner^{d}\triangle$ B A C, and the Perpendicular B D let fall from the right Angle A B C upon the Hypothenuſe A C being given, thence to find the Hypothenuſe A C.

Solution.

Let	1	$d = AE = 15$
and	2	$p = BD = 36$
ſuppoſe	3	$a = AC = ?$
and	4	$e = EB$
4. 6. *Eucl. El.*	5	$d + e\ (AB)\ .\ .\ p\ (BD)\ ::\ a\ (AC)\ .\ .\ BC = e$
Æq.	6	$de + ee = pa$
47. 1. *Eucl. El.*	7	$dd + 2de + ee + ee\ (= ABq : + BCq :) = dd + 2de + 2ee = aa$
6×2	8	$2de + 2ee = 2pa$
$7 - 8$	9	$dd = aa - 2pa$. Caſe 2.
$9 + pp$	10	$dd + pp = aa - 2pa + pp$
10ω2	11	$\sqrt{dd + pp} = a - p$
$11 + p$	12	$p + \sqrt{dd + pp} = a = 75$

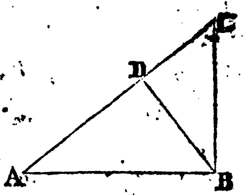
Conſtruction.



With L A $= 2p$, and A E $= d$, make the $\llcorner^{d}$ ▭ L A E N, and upon M, the middle Point of L A, deſcribe ſuch a Semicircle as will paſs thro' N and E, the remoteſt Points of the ▭; then compleat the Diameter, by producing L A to O and C; and C A = L O will be $= a$; for C L $(a - 2p)$ . . L N (d) :: L N (d)

∴ L O = a, *per* 13. 6. *Eucl. El.* Whence $aa - 2pa = dd$; as in the 9th Step.

PROBLEM III.

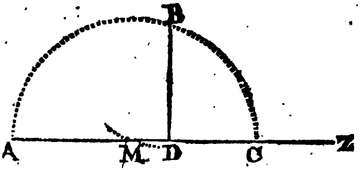


The Hypothenuſe A C of any ∟d △ A B C, and the Perpendicular B D, let fall from the ∟ A B C, upon the Hypothenuſe A C being given, to find A D, the greater Segment of the Hypothenuſe.

Solution.

Let	1	$b = AC = 75$
	2	$p = BD = 36$
ſuppoſe	3	$AD = a = ?$
By 4. 6. *Eucl. El.*	4	$a \,..\, p :: p \,..\, DC$
therefore	5	$\frac{pp}{a} = DC$
$1 - a$	6	$b - a = DC$
$5 = 6$	7	$\frac{pp}{a} = b - a$
$7 \times a$	8	$pp = ba - aa.$ Caſe 3.
8 tranſp.	9	$aa - ba = - pp$
$9 + \frac{bb}{4}$	10	$aa - ba + \frac{bb}{4} = \frac{1}{4}bb - pp$
10ww2	11	$a - \frac{1}{2}b = \pm\sqrt{\frac{1}{4}bb - pp}$
$11 + \frac{1}{2}b$	12	$a = \frac{1}{2}b \pm \sqrt{\frac{1}{4}bb - pp} = 48$ or 27

The Geometrical Conſtruction of Caſe 3. *viz.* $ba - aa = pp$, may be thus perform'd.



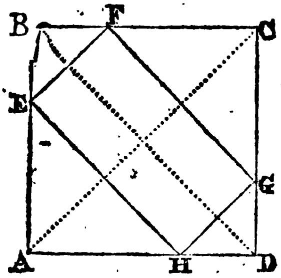
Draw a right Line of any convenient Length, as A Z, and near its Middle erect a Perpendicular D B = p: from the Top, or upper End B of that Perpendicular ſet off B M = $\frac{1}{2}b$; and upon the Point M, where B M touches A Z with the Radius M B, deſcribe a Semicircle A B C M; then will its Diameter A C = b be cut by the Perpendicular B D into two Segments, A D and D C, which are the two Values of the Root a, *viz.* the greater and leſſer Roots, the Sum of both being always = to the Coefficient b.

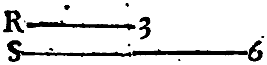
For $DG\ (b-a)\ ..\ DB\ (p)\ ::\ DB\ (p)\ ..\ DA = a$, *per* 13. 6. *Eucl. El. Ergo*, $ba - aa = pp$, as above.

CHAP. IV.

The Solution of PROBLEMS *of ſeveral Sorts.*

PROBLEM I.





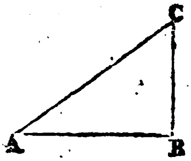
Within a given Square A B C D, to inſcribe a long Square E F G H, whoſe Sides E F, or G H, ſhall be in Proportion to E H, or F G as R to S, and parallel to the Diameters A C, B D of the Square.

Solution.

Let	1	$b = AB = AD = 12$
Suppoſe	2	$a = BF = BE = HD = DG$
Then	3	$FC = b - a = EA$
By 47. 1. *Eucl. El.*	4	$\sqrt{2aa}\ (= \overline{EBq : + BFq}\vert^{\frac{1}{2}}) = EF = a\sqrt{2}$
	5	$\overline{b - a} \times \sqrt{2}\ (= \overline{FCq : + CGq}\vert^{\frac{1}{2}}) = FG$
By the Prob.	6	$R\ ..\ S\ ::\ a\sqrt{2}\ ..\ \overline{b - a} \times \sqrt{2}$
conſeq.	7	$R\ ..\ S\ ::\ a\ ..\ b - a$
Æquat.	8	$Rb - Ra = Sa$
$8 + Ra$	9	$Rb = Ra + Sa$
$9 \div \overline{R + S}$	10	$\frac{Rb}{R + S} = a = 4$

The

PROBLEM II.



The Perimeter, (*viz.* the Sum of the 3 Sides, A B, A C, B C) of any ∟d △ A B C, and its Area being given; thence to find each Side.

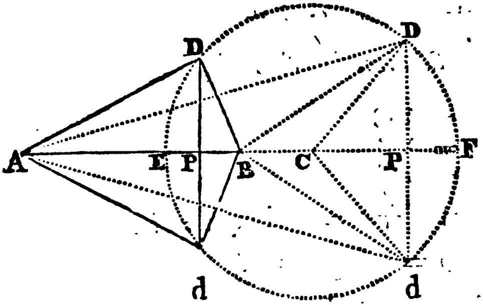
Solution.

Let	1	$a = AC$
	2	$e = AB$
	3	$y = BC$
then by the Prob.	4	$a + e + y =$ Perimeter $= b = 12$
	5	$\frac{1}{2} ey =$ the Area $= 6$
By 47. 1. *Eucl. El.*	6	$yy + ee = aa$
5×4	7	$2ey = 4c$
$6 + 7$	8	$yy + 2ey + ee = aa + 4c$
$4 - a$	9	$e + y = b - a$
9 @ 2	10	$e^2 + 2ey + yy = b^2 - 2ba + aa$
$8 = 10$	11	$a^2 + 4c = b^2 - 2ba + a^2$
$11 - a^2 - 4c + 2ba$	12	$2ba = b^2 - 4c$
$12 \div 2b$	13	$a = \frac{b^2 - 4c}{2b} = 5$
From 9 and 13	14	$e + y = b - \frac{\overline{bb - 4c}}{2b} = \frac{b^2 + 4c}{2b}$
$6 - 7$	15	$e^2 - 2ey + yy = aa - 4c$
From 13 and 15	16	$e^2 - 2ey + y^2 = \frac{b^4 - 24b^2c + 16cc}{4bb}$
16 ɯ 2	17	$e - y = \frac{\pm\sqrt{b^4 - 24b^2c + 16cc}}{2b}$
$14 + 17$	18	$2e = \frac{b^2 + 4c \pm \sqrt{b^4 - 24b^2c + 16c^2}}{2b}$
$18 \div 2$	19	$e = \frac{b^2 + 4c \pm \sqrt{b^4 - 24b^2c + 16c^2}}{4b} = \begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$
$14 - 17$	20	$2y = \frac{b^2 + 4c \mp \sqrt{b^4 - 24b^2c + 16c^2}}{2b}$
$20 \div 2$	21	$y = \frac{b^2 + 4c \mp \sqrt{b^4 - 24b^2c + 16cc}}{4b} = \begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$

PRO-

PROBLEM III.

(*Apol. Perg.*)



Two Points, AB in a Plane, being given, ſo to deſcribe in the ſame Plane a Circle D D, that to every Point of its Circumference D the ſtraight Lines AD, BD being drawn, ſhall be in a given Proportion of *m* (the greater) to *n* (the leſſer.)

Inquiſition.

Suppoſe it done, and C the Center of the Circle: Then, becauſe the Points A, B are given, the Line A B is given; and becauſe of *m* to *n*, its Point E, this being one of the Points D; therefore A E and B E are given.

Put A E $= b$, then $m \,..\, n :: b \,..\, \frac{n}{m} b =$ BE. Suppoſing then from D on AB produc'd, if Need be, a $\perp$ DP, and taking (on oppoſite Sides of AB) AD $=$ Ad, and BD $=$ Bd; the $\triangle$s on the oppoſite Sides of A B will be ſimilar and equal; and therefore DPd one ſtreight Line, biſected in P, and at $\llcorner$s to AB; In which therefore is the Center C, and the Diameter ECF, which muſt lie towards the Side B, not A, otherwiſe B F would be $\sqsubset$ AF, that is here BD $\sqsubset$ A D.

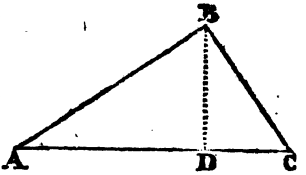
Now ſuppoſe EC $= r$, and E$p = x$; then A$p = b + x$, and B$p = \frac{n}{m} b \backsim x$; then (by the Property of a Circle) $\underline{2rx - xx} = \underline{PDq : \text{ And}}$ (by the 47. 1. *Eucl. El.*) $2rx - xx + bb + 2\,2bx + xx = ADq : = 2rx + 2bx + bb$: As alſo $2rx - xx + \frac{n^2}{m^2} b^2 - \frac{2n}{m} bx + xx = BDq = 2rx - \frac{2n}{m} bx + \frac{n^2}{m^2} b^2$.

Now (by *Hyp.*) $m \,..\, n :: AD \,..\, BD$, conſequently $m^2 \,..\, n^2 :: ADq \,..\, \underline{\underline{BDq}}$: that is $:: 2rx + 2bx + bb \,..\, 2rx - \frac{2n}{m} bx + \frac{n^2}{m} b^2$; wherefore $2m^2rx - 2nmxb + n^2 b^2 = 2n^2rx + 2n^2bx + n^2 b^2$, and $m^2r - nmb = n^2r + n^2b$.

and by Tranſpoſition $m^2r - n^2r = mnb + n^2b$, and by dividing each Part of the laſt Step by $m + n$ you'll have $\overline{m - n} \times r = nb$.

And laſtly, by dividing each Part by $\overline{m - n}$, you'll have $r = \frac{nb}{m - n}$

PROBLEM IV.

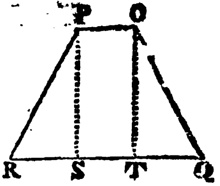


$\angle ABC = \llcorner = \llcorner ADB$ } *Data.*
$AB = b$
$DC = c$
$AD = a$ *Quæſit.*

Solution.

The $\triangle$s A B D and D B C, are ſimilar;
Wherefore $a \,..\, DB :: DB \,..\, c$
2. Therefore $ac = DBq$:
3. By 47. 1. *Eucl. El.* $bb - aa = ac$
4. By Tranſp. $bb = aa + ac$
5. By Comp. $\square$: $bb + \frac{1}{4} cc = aa + ac + \frac{1}{4} cc$
6. By. Evol. $\overline{bb + \frac{1}{4} cc}|^{\frac{1}{2}} = a + \frac{1}{2} c$
7. By Tranſp. $\overline{bb + \frac{1}{4} cc}|^{\frac{1}{2}} - \frac{1}{2} c = a$ Canon.

LEMMA to PROBLEM V.



The Area of any Trapezium P O Q R, whoſe oppoſite Sides O P and Q R are parallel, is had by multiplying the Sum of the Sides O P and Q R by half the common Height S P or T O.

Suppoſe $PS = OT = b$, $RQ = l$, $PO = ST = m$, and let RS be $= n$; then TQ will be $= l - m - n$.

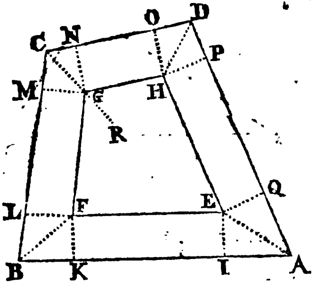
I ſay, $\overline{l+m}\times\frac{1}{2}b=$ Area of Trapezium O P R Q.

By 41. 1. *Eucl. El.*

$$\left\{\begin{array}{c} n \\ \overline{l-m-n} \\ 2m \end{array}\right\}\times\tfrac{1}{2}b\left\{\begin{array}{l} =\text{Area of }\triangle RSP. \\ =\text{Area of }\triangle TOQ. \\ =\text{Area of }\square POST. \end{array}\right.$$

And by adding the three foregoing Areas into one Sum you'll have $\overline{n+l-m-n+2m}\times\frac{1}{2}b=\overline{l+m}\times\frac{1}{2}b$ = Area of the Trapezium P O Q R. Q E D.

PROBLEM V.



The Area of a Trapezium and its Sides, ſeverally being given, to make a Trapezium within the former, in ſuch Manner that the Sides of the one may be every where ſeparated from the Sides of the other, by an equal parallel Diſtance; and that the Space lying between both Trapezia may be equal to any poſſible given Space.

$$\left.\begin{array}{l} \text{The Sides of the given Trapezium,}\left\{\begin{array}{l} AB=b\\ BC=c\\ CD=d\\ DA=f \end{array}\right.\\ \text{The Area of the Trapezium } ABCD=s\\ \text{The Area of the Space lying between both Trapezia,}\ \}=t\\ \text{Then the Area of the inſcrib'd Trapezium } EFGH,\ \}=s-t \end{array}\right\}\textit{Data.}$$

The equal parallel Diſtance $=a=?$

Preparation.

Preparation.

Suppoſe EFGH to be the Trapezium requir'd to be inſcrib'd in the Trapezium given ABCD; then the Space lying between both Trapezia is = to the Sum of the Trapezia

$$\left.\begin{array}{c} AEFB \\ BFGC \\ CGHD \\ DHEA \end{array}\right\} = t$$

Draw EI and FK $\perp$s BA, FL and GM $\perp$s BC, GN and HO $\perp$s CD, and HP and EQ $\perp$s DA; then a is = EI = FK = FL = GM = GN = HO = HP = EQ.

Solution.

The Trapezia ABCD and EFGH are ſimilar (for CG being produc'd to R makes the $\angle$HGR = $\angle$DCG and $\angle$FGR = $\angle$BCG (by 29. 1. *Euc. El.*) conſequently the $\angle$HGF is = $\angle$DCB; and for the like Reaſon the $\angle$GHE is = $\angle$CDA; alſo $\angle$HEF is = $\angle$DAB, and $\angle$EFG = $\angle$ABC; therefore (by the 20. 6. *Eucl. El.*)

$$s \,..\, s - t :: b^2 \,..\, \frac{b^2 s - b^2 t}{s} = \mathrm{EF}q:$$

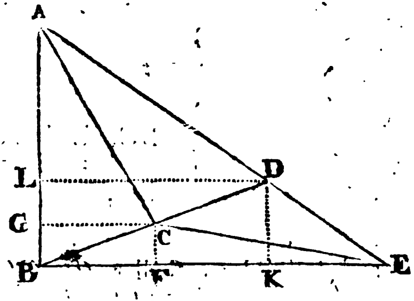
therefore $\sqrt{\frac{bbs - bbt}{s}} = \mathrm{EF} = b\sqrt{\frac{s-t}{s}}$.

And for the like Reaſons,
$$\left\{\begin{array}{l} c\sqrt{\frac{s-t}{s}} = \mathrm{FG} \\ d\sqrt{\frac{s-t}{s}} = \mathrm{GH} \\ f\sqrt{\frac{s-t}{s}} = \mathrm{HE} \end{array}\right.$$

Wherefore by the preceding Lemma,
$$\left\{\begin{array}{l} +b+b\sqrt{\frac{s-t}{s}} \\ +c+c\sqrt{\frac{s-t}{s}} \\ +d+d\sqrt{\frac{s-t}{s}} \\ +f+f\sqrt{\frac{s-t}{s}} \end{array}\right\} \times \tfrac{1}{2}a = t$$

Conſequently $a = \dfrac{2t}{b+c+d+f+\overline{b+c+d+f} \times \overline{\frac{s-t}{s}}\big|^{\frac{1}{2}}}$

PROBLEM VI.



The Angle ABE being right, the ∠CBE and Side CB being given, as alſo CD = CB; it is requir'd to draw the ſtraight Line ADE thro' the Extremity of the given Line BCD, ſo as the Lines drawn from C to the Angles A and E may be equal, *viz.* AC = CE.

Analyſis.

1. Suppoſe it done, and thro' C draw CG ∥ EB and CF ∥ AB.

2. Thro' D likewiſe draw DL ∥ EB, and DK ∥ AB.

3. In the △ BCF you have the ∠s B and F, and the Side CB given, and therefore the Sides BF and CF are eaſily found by Trigonometrical Calculation; wherefore ſuppoſe them known, and put CF $= b$, and BF $= d$.

4. Put BC $= c$, and therefore BD $= 2c$ then

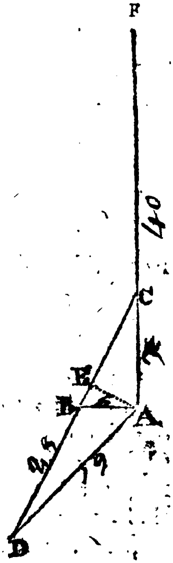
By 4. 6. *Eucl. El.*	5	LD = BK = $2d$, and DK = LB = $2b$
Suppoſe	6	KE = a = ?
therefore	7	FE = $d + a$
By 47. 1. *Eucl. El.*	8	$\square$CE = $\square$FE + $\square$CF $= d^2 + 2ad + aa + bb$
By 4. 6. *Eucl. El.*	9	a (KE) . . $2b$ (KD) : : $2d$ (DL) . . $\frac{4db}{a}$ = AL
	10	AG = AL + LG $= \frac{4db}{a} + b = \frac{4db + ba}{a}$
By 47. 1. *Eucl. El.*	11	$\square$AC = $\square$AG + $\square$GC $= \frac{16d^2b^2 + 8db^2a + b^2a^2 + ddaa}{aa}$

Now by Supp. $\overline{8}|^{\frac{1}{2}} = \overline{11}|^{\frac{1}{2}}$ and theref. $8 = 11$ | 12 | $d^2 + 2ad + aa + b^2 = \frac{16d^2b^2 + 8db^2a + b^2a^2 + d^2a^2}{aa}$

12 reduc'd | 13 | $a^4 + 2a^3d - 8db^2a = 16d^2b^2$

PROBLEM VII.

A Tree AF, *b* Foot high, ſtanding upon the Side of a Hill, was by a tempeſtuous Wind broke in a Point C; the upper Part of it CF fell ſo as to become CD; and the Diſtance from D its Top to A its Root was found to be *c* Foot; and a Horizontal Line AB being drawn 'till it cut the Part CD of the Tree in B, was found to be *d* Foot. It is requir'd to tell how many Foot long the ſtanding Part AC is.



In order to ſolve this Problem, I ſhall ſuppoſe the Tree AF to be perpendicular to the Horizon, and then in the △ ACD, I have DC + CA = *b*, AD = *c*, Angle CAB right, and AB = *d* given; to find CA = *a* = ?

Let fall AE ⊥ DC, ſo is DC divided into two Segments DE and EC.

Solution:

By Axiom 4. of *Pl. Trig.* | 1 | $b - a \,.\, c + a :: c - a \,.\, \frac{cc - aa}{b - a} = DE - EC$

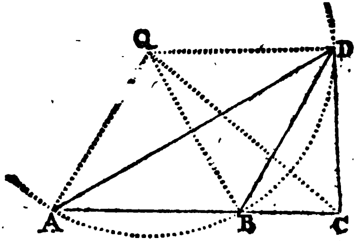
2 | $\frac{b - a}{2} - \frac{cc - aa}{2b - 2a} = \frac{bb - 2ba + 2aa - cc}{2b - 2a} = EC$

47. 1. *Eucl. El.* | 3 | $BC = \overline{aa + dd}|^{\frac{1}{2}}$ but BD = DA

4. 6. *Eucl. El.* | 4 | $\sqrt{a^2 + d^2} \,.\, a :: a \,.\, \frac{aa}{\sqrt{a^2 + d^2}} = EC$

$$\begin{array}{r|l|l} 2=4 & 5 & \dfrac{b^2-2ba+2aa-cc}{2b-2a}=\dfrac{aa}{\sqrt{a^2+d^2}} \\ 5 \text{ reduc'd} & 6 & \left.\begin{array}{l} \overline{+8b^2-4c^2+4d^2-4b^2}\times a^4 \\ \overline{-4b^3+4bc^2-8d^2b}\times a^3 \\ \overline{+b^4-2b^2c^2+c^4+8d^2b^2-4d^2c^2}\times a^2 \\ \overline{-4d^2b^3+4d^2bc^2}\times a \end{array}\right\} = \left\{\begin{array}{l} -d^2b^4 \\ +2d^2b^2c^2 \\ -d^2c^4 \end{array}\right. \end{array}$$

PROBLEM VIII.



In the annex'd △
$\left.\begin{array}{l} AB=2 \\ BC=1 \end{array}\right\}$ that is, $AC=3$
$CD=\sqrt{3}$
$\angle ADB=30°:00'$; Its Sine (the Radius being 10) $=5$; Its Co-ſine (that is, the Sine of $60°:00'$) $=\sqrt{100-25}=\sqrt{75}$

Requir'd to find $AD=a=$?

Conſtruction.

On $AB=2$ make an Iſoſceles △ AQB, whoſe $\angle AQB$ ſhall be $=2\ \angle$s $ADB=60°$ [thus ſubtract $2\ \angle$s $ADB=60°$ from $180°$, and half the Remainder is $=\angle QAB=\angle QBA$]; on Q, as a Center, deſcribe the Circle ABD paſſing thro' A and B. Produce the Line AB to C, making $BC=1$; then placing one Foot of your Compaſſes in the Point C, with the other mark the Point D in the Circumference of the ſaid Circle ABD, making $CD=\sqrt{3}$. Draw DC, DA, and DB, and the △ is conſtructed.

Method of Calculation.

Draw QD and QC.

1. In the △ AQB you have the Side AB given, and the $\angle AQB$ is $=2\ \angle$s ADB (by 20. 3. *Eucl. El.*) $=60°$ is given; therefore $\angle QAB=\angle QBA$ (by 5. 1. *Eucl. El.*) is given; for $\angle$s $QAB+QBA$ are $=$ Complement of $\angle AQB$ to $180°$ (by 32. 1. *Eucl. El.*); therefore you may find (by the 2d Axiom of Plain Trig.) $BQ=QA$.

2. Then in the △ AQC you have the Sides AQ and AC and the $\angle QAC$ given, by which you may find (by *Axiom* 3. *Pl. Trig.*) the Side QC,

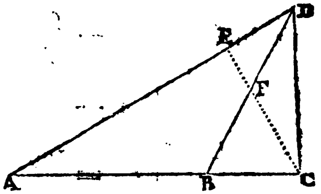
3. Now in the △ QDC you have the Sides QD = QA (by the Definition of a Circle), QC and CD given; therefore you may (by the 4th *Axiom of Pl. Trig.*) find the ∠s CQD, QCD, QDC; then,

4. To the ∠CQD add the ∠CQB, and the Sum is = ∠DQB = 2 ∠s DAB (by 20. 3. *Eucl. El.*); whence ∠ DAB is given, by the Help of which you may find DA, *&c.*

5. Or to the ∠ QCB add the ∠QCD, and the Sum is = ∠ DCB; So you have DC, AC, and ∠ DCB or ∠ DCA given; consequently (by the 3d Axiom of *Pl. Trig.*) you may find the Side AD; and so you are put in a Way of finding the rest.

An Algebraical Solution of Problem 8.

On AD let fall the Perpendicular CFE.



Axiom 1. *Plain Trigonometry.*	1	$AB \,..\, S\angle ADB :: AD \,..\, S\angle ABD = S,\angle CBD$ $2 \,:\, 5 :: a :: \frac{5a}{2}$
Axiom 4 *Plain Trigonometry.*	2	$AD \,..\, AC + DC :: AC - DC \,..\, AE - ED$ $a \,..\, a + \sqrt{3} :: a - \sqrt{3} \,..\, \frac{9-3}{a} = \frac{6}{a}$
	3	$\frac{1}{2} AD + \frac{1}{2} \overline{AE - ED} = AE$ $\frac{a}{2} + \frac{6}{2a} = \frac{aa+6}{2a}$
	4	$\frac{1}{2} AD - \frac{1}{2} AE + \frac{1}{2} ED = BD$ $\frac{a}{2} + \frac{6}{2a} = \frac{aa-6}{2a}$
Axiom 1 *Plain Trigonometry.*	5	$S\angle EFD \,..\, ED :: S\angle EDF \,..\, EF$ $\sqrt{75} \,..\, \frac{aa-6}{2a} :: 5 \,..\, \frac{5aa-30}{2a\sqrt{75}}$

The ∠EFD is = ∠BFC (by 15. 1. *Eucl. El.*); therefore ∠BFC is = $\sqrt{75}$

Axiom 2 *Plain Trigonometry.* 6 $$\begin{cases} S\angle BFC \;..\; BC :: S,\angle CBF \;..\; CF \\ \sqrt{75} \;..\; 1 :: \frac{5a}{2} \;..\; \frac{5a}{2\sqrt{75}} = \\ \frac{5aa}{2a\sqrt{75}} \end{cases}$$

7 $$\begin{cases} EF + FC = EC \\ \frac{5a^2-30}{2a\sqrt{75}} + \frac{5a^2}{2a\sqrt{75}} = \frac{10a^2-30}{2a\sqrt{75}} \end{cases}$$

By 47. 1. *Eucl. El.* 8 $$\begin{cases} ECq + AEq = ACq: \\ \frac{100a^4-600a^2+900}{4aa\times75} + \frac{a^4+12a^2+36}{4a^2} = 9 \end{cases}$$

8th reduc'd 9 $$aaaa - \frac{96}{7}aa = -\frac{144}{7} = \frac{1008}{49}$$

Comp. □ 10 $$\begin{cases} a^4 - \frac{96}{7}a^2 + \frac{2304}{49} = -\frac{1008}{49} + \frac{2304}{49} \\ = \frac{1296}{49} \end{cases}$$

10ω2 11 $$a^2 - \frac{48}{7} = \pm\sqrt{\frac{1296}{49}} = \pm\frac{36}{7}$$

11 + $\frac{48}{7}$ 12 $$a^2 = \frac{48}{7} \pm \frac{36}{7} = \frac{84}{7} \text{ or } \frac{12}{7} = 12 \text{ or } \frac{12}{7}$$

10ω2 13 $$a = \sqrt{12} \text{ or } = \sqrt{\frac{12}{7}}$$

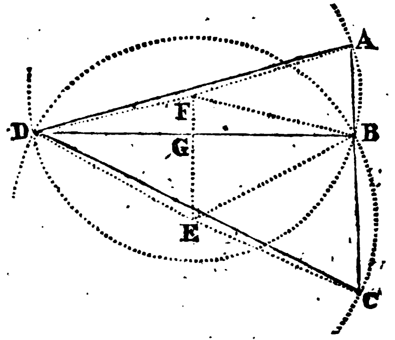
PROBLEM IX.

$$\left.\begin{array}{l} \left.\begin{array}{l} AB = 6 \\ BC = 11 \end{array}\right\} \text{that is } AC = 17 \\ \left.\begin{array}{l} \angle ADB = 15^\circ \\ \angle BDC = 25^\circ \end{array}\right\} \text{that is, } \angle ADC = 40^\circ \end{array}\right\} \text{being given,}$$

Requir'd to find the other Sides and Angles.

Construction.

Conſtruction.



On BC = 11 make an Iſoſceles △ EBC; whoſe ∠ BEC = 2 ∠s BDC = 50°.

On E, as a Center, deſcribe the Circle CBD, paſſing thro' B and C.

On AB = 6 make an Iſoſceles △ AFB, whoſe ∠ AFB is = 2 ∠s ADB = 30°. On F as a Center deſcribe a Circle BAD, paſſing thro' A and B, and cutting the former Circle in D. Draw AD, BD, and CD, and the △ is conſtructed.

Method of Calculation.

Draw DE, and DF, and FE, cutting DB in G.

1. EB = EC (by the Definition of a Circle); therefore (by 6. 1. *Eucl. El.*) ∠ EBC = ∠ ECB will be given; for the Sum of them both is = Complement of the ∠ BEC to 180°. And BC being given EC = EB is given; for it may be found by Axiom 2 of *Pl. Trig.* The ſame Way AF = FB will be found, and ∠ FAB = ∠ FBA.

2. If from the ∠ EBA = Compliment of ∠ EBC to 180°, 00′ (by 13. 1. *Eucl. El.*) you take the ∠ ABF, the Remainder ∠ FBE will be given.

3. In the △ FBE you have the Sides FB and BE, as alſo the ∠ FBE given, by which (by Axiom 3 *Pl. Trig.*) you may find the Side FE and the ∠s BFE and FEB.

4. The △s DFE and BFE have the Side FE common to both, and the Side DF = FB, as alſo the Side DE = EB; therefore (by 8. 1. *Eucl. El.*) ∠ BFG = ∠ DFG; therefore (by 10. 1. *Euc. El.*) the Sides DG and GB are equal; and (by 11. 1. *Eucl. El.*) GF and GE are ⊥s DB.

5. Now FB being already found, and the ∠ BFE being known, as alſo ∠ FGB = 90°, the Compliment of the Sum of them both to 180°, *viz.* ∠ GBF will be given, and conſequently GB = GD = ½ DB; for it may be found by Axiom 1 of *Plain Trig.* and ſo you are put in a Way of finding the reſt.

An

An Algebraical Solution of Problem 9.

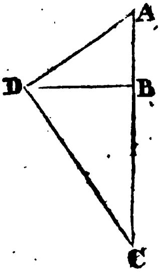
$$\left.\begin{array}{l}\left.\begin{array}{r}\text{Let AB} = b\\ \text{BC} = c\end{array}\right\}\text{that is, AC} = b + c\\ \left.\begin{array}{r}\text{S, }\angle\text{ADB} = f\\ \text{S, }\angle\text{BDC} = g\end{array}\right\}\text{and }\Sigma\ \angle\text{ADC} = m\end{array}\right\}\text{given,}$$

Requir'd to find DA = a = ?

Let AD be $\sqsupset$ AC;

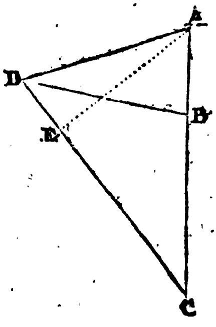
Then the Problem admits of three Caſes; that is, the given ∠ ADC is equal to, or leſs, or greater than a Right Angle.

Solution of Caſe 1. *Viz.* When ∠ ADC = ∟



Axiom 2 *Pl. Trig.*	1	$b \,..\, f :: a \,..\, \frac{fa}{b} = \text{S, }\angle\text{ABD} = \text{S}\angle\text{DBC}$
	2	$g \,..\, c :: \frac{fa}{b} \,..\, \frac{cfa}{bg} = \text{DC}$
47 1 *Eucl. El.*	3	$\frac{ccffaa}{b^2g^2} + a^2 = b^2 + 2bc + c^2$
3d reduc'd	4	$a = \sqrt{\frac{b^4g^2 + 2b^3cg^2 + b^2c^2g^2}{c^2f^2 + b^2g^2}}$

Caſe 2. When A D C = an acute Angle.
Let fall A E ⊥ DC.



Axiom 2 *Pl. Trig.*	1	$b \,..\, f :: a \,..\, \frac{fa}{b} = \text{S, }\angle\text{DBA} = \text{S}\angle\text{DBC}$
By ſaid Ax.	2	$g \,..\, c :: \frac{af}{b} \,..\, \frac{afc}{gb} = \text{DC}$
By Axiom 4 of *Plain Trigonom.*	3	$\frac{cfa}{gb} \,..\, a + \overline{b+c} :: \overline{b+c} - a \,..\, \frac{-bga^2 + b^3g + 2b^2cg + c^2bg}{cfa} = \text{EC} - \text{ED}$

4 *cfa*

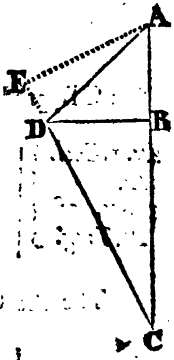
4 $\left\{ \frac{cfa}{2gb} - \frac{-bga^2 + b^3g + 2b^2cg + c^2bg}{2cfa} = \text{DE the leſſer Segment.} \right.$

Ax. 1. *Pl. Trig.* 5 $\left\{ m \,.\,.\, \frac{cfa}{2gb} - \frac{-bga^2 + b^3g + 2b^2cg + c^2bg}{2cfa} :: {}^{*}r \,.\,.\, a \right.$ (* r = Radius.)

Æq. 6 $\left\{ ma = \frac{rcfa}{2gb} - \frac{-rbga^2 + rb^3g + 2rb^2cg + rc^2bg}{2cfa} \right.$

6 reduc'd 7 $a = \overline{\frac{rb^4g^2 + 2rb^3g^2c + rb^2c^2g^2}{rc^2f^2 + rb^2g^2 - 2mbgcf}}\Big|^{\frac{1}{2}}$

Caſe 3. When ∠ ADC = an obtuſe Angle, let fall AE ⊥ CD produc'd.



By Ax. 2. *Pl. Trig.* 1 $\left\{ b \,.\,.\, f :: a \,.\,.\, \frac{fa}{b} = \text{S}\angle\text{ABD} = \text{S}\angle\text{DBC} \right.$

By ſaid 2d Ax. 2 $g \,.\,.\, c :: \frac{af}{b} \,.\,.\, \frac{cfa}{gb} = \text{DC}$

By 12. 2. *Eucl. El.* 3 $\left\{ \overline{\frac{b^2+2bc+c^2}{2} - \frac{a^2}{2} - \frac{c^2f^2a^2}{2b^2g^2}} \div \frac{caf}{bg} = \text{ED} = \frac{b^3g + 2b^2cg + c^2bg - a^2bg}{2cfa} - \frac{cfa}{2bg} \right.$

Ax. 1. *Pl. Trig.* 4 $\left\{ m \,.\,.\, \frac{b^3g + 2b^2cg + c^2bg - a^2bg}{2cfa} - \frac{cfa}{2bg} :: r \,.\,.\, a \right.$

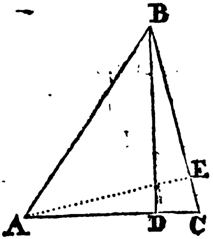
Æq. 5 $\left\{ ma = \frac{rb^3g + 2rb^2cg + rc^2bg - ra^2bg}{2cfa} - \frac{rcfa}{2bg} \right.$

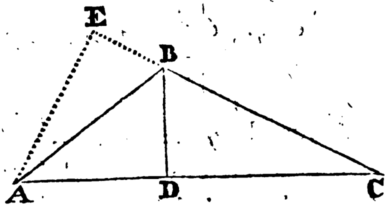
5 reduc'd 6 $a = \sqrt{\frac{rb^4g^2 + 2rb^3g^2c + rb^2c^2g^2}{rc^2f^2 + rb^2g^2 + 2mbgcf}}$

PROBLEM X.

In the annex'd △s
AB = b
DC = c and ⊥ DB; and ∠ ABC * either acute or obtuse, and its Sine = p } given;
DA = a requir'd.

* *When* ∠ ABC = ∟. *See Prob.* 4.





Solution.

On CB, produc'd, if Need be, let fall the Perpendicular AE.

47. 1. *Eucl. El.*	1	$\sqrt{\overline{bb - aa}} = DB$
1 Ax. of *Pl. Trig.*	2	$r \,..\, b :: p \,..\, \frac{pb}{r} = AE$

The △s CBD and CAE are similar. Wherefore

4. 6. *Eucl. El.*	3	$\sqrt{\overline{b^2 - a^2}} \,..\, \frac{pb}{r} :: c \,..\, \frac{cpb}{r\sqrt{\overline{b^2 - a^2}}} = EC$
47. 1. *Eucl. El.*	4	$\frac{p^2b^2}{r^2} + \frac{c^2p^2b^2}{rr \times \overline{b^2 - a^2}} = ACq := aa + 2ac + cc$ *Æquat.*

LEMMA

LEMMA I. to PROBLEM XI.

Fig. 1. In which DE is ⊐ Quadrant.

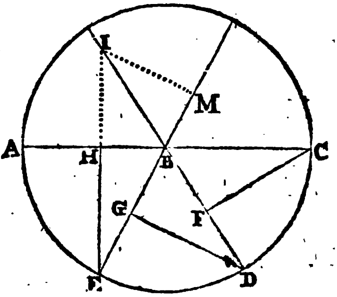
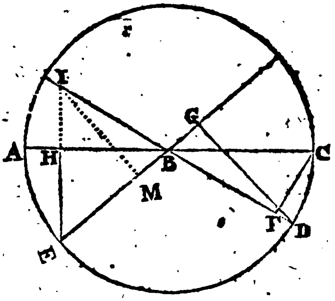


Fig. 2. In which DE is ⊏ Quadrant.



Two Arcs CE and CD being given, the greater of which CE is more, and the other CD leſs, than a Quadrant; if you divide by *Radius* the Difference of the Rectangles of their Sines and Co-ſines, you have the Co-ſine of their Difference DE.

Sine of CE is EH $= b$, its Co-ſine is HB $= m$.
Sine of CD is CF $= c$, its Co-ſine is BF $= n$.
The Coſine of DE is GB $= a$?
I ſay, that $\frac{cb \backsim mn}{r} = a$ (N. B. $r =$ *Radius.*)

Demonſtration.

Produce EH 'till it meets BD produc'd in I, and from I let fall IM $\perp$ EB produc'd.

The $\llcorner$d $\triangle$s BCF and BIH are ſimilar, (for $\angle$B is equal both) therefore

$$\left.\begin{matrix}\text{By 4. 6.}\\ \textit{Eucl. El.}\end{matrix}\right\}\left|\begin{array}{l|l} 1 & n\,.\,.\,m :: \left\{\begin{array}{l} r\,.\,.\,\frac{mr}{n} = \text{BI} \\ c\,.\,.\,\frac{mc}{n} = \text{HI} \end{array}\right. \\ 2 & \text{EH}+\text{HI} = \text{EI} = b + \frac{cm}{n} = \frac{bn+cm}{n} \end{array}\right.$$

The $\llcorner$d $\triangle$s BEH and MIE are ſimilar; therefore

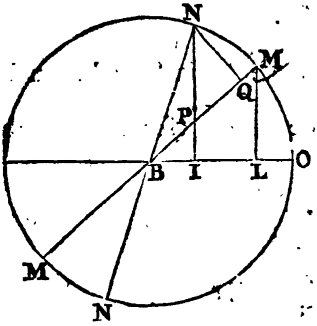
By 4. 6. Eucl. El.	3	$r . b :: \frac{bn + cm}{n} . \frac{bbn + bcm}{rn} = \mathrm{EM}$
	4	$\mathrm{EM} \backsim \mathrm{EB} = \mathrm{BM} = \frac{\frac{b^2n + bcm}{rn} \backsim r}{} = \frac{r^2n \backsim \overline{b^2n + bcm}}{rn} = \frac{\overline{r^2 - b^2} \times n \backsim bcm}{rn}$
47. 1. Eucl. El.	5	$r^2 - b^2 = m^2$
From 4 and 5	6	$\frac{mmn \backsim bcm}{rn} = \mathrm{BM}$

The $\triangle$s BIM and BDG are ſimilar; wherefore

By 4. 6. Eucl. El.	7	$\frac{mr}{n} . \frac{bcm \backsim m^2n}{rn} :: r . a$
Æq.	8	$a = \frac{cb \backsim mn}{r}$; that is

When the Difference DE of the given Arcs is leſs than a Quadrant; (which is the Caſe that concerns our Problem 11.) then a is $= \frac{cb - nm}{r}$; but when DE is greater than a Quadrant, then $a = \frac{mn - cb}{r}$. W. W. D.

LEMMA II.



Two Arches NO and MO being given, the greater of which NO is leſs than a Quadrant, (or the leſs of which NO is more than a Quadrant;) if you divide by *Radius* the Difference of the Rectangles of the Sine of the one multiply'd by the Co-ſine of the other, the Quotient will be the Sine of MN their Difference.

Sine of NO is NI $= b$, its Co-ſine is BI $= m$

Sine of MO is ML $= c$, its Co-ſine is BL $= n$,

Sine of MN is NQ $= a$.

I ſay, that $\frac{bn-cm}{r}=a$

Demonſtration.

The Δs BLM, BIP, and QPN are alike; therefore

4. 6. *Eucl. El.*	1	$n \,..\, c :: m \,..\, \frac{mc}{n} = IP$
	2	$NP = NI - IP = b - \frac{mc}{n} = \frac{bn-mc}{n}$
4. 6. *Eucl. El.*	3	$r \,..\, n :: \frac{bn-mc}{n} \,..\, \frac{bn-mc}{r} = a$

Q. E. D.

LEMMA III. *See Fig. to Lemma 2.*

Two Arches NO and MO, the greater of which NO is leſs (or the leſs of which NO is greater) than a Quadrant, being given, if you divide by *Radius* the Sum of the Rectangles of their Sines and Co-ſines, you have the Co-ſine of their Difference NM.

Sine of NO is NI $= b$, its Co-ſine is BI $= m$.
Sine of MO is ML $= c$, its Co-ſine is BL $= n$;
Co-ſine of NM is BQ $= a$.
I ſay, that $\frac{bc+mn}{r}=a$

Demonſtration.

The Δs BML, BPI, and PNQ are ſimilar; therefore

4. 6. *Eucl. El.*	1	$n \,..\, m :: \begin{cases} r \,..\, \frac{rm}{n} = BP \\ c \,..\, \frac{cm}{n} = IP \end{cases}$
	2	$NP = NI - IP = b - \frac{cm}{n} = \frac{bn-cm}{n}$
	3	$r \,..\, c :: \frac{bn-cm}{n} \,..\, \frac{cbn-c^2m}{rn} = PQ$

4 $$\begin{cases} BP + PQ = BQ = \frac{rm}{n} + \frac{cbn - c^2m}{rn} \\ = \frac{r^2m - c^2m + cbn}{rn} = a \end{cases}$$

47.1 *Eucl. El.* 5 $$r^2 - c^2 = nn$$

from 4 and 5 6 $$\frac{n^2m + cbn}{rn} = a = \frac{nm + cb}{r}$$

W. W. D.

L E M M A IV. *See Figures to Lemma* 1.

Two Arches CE and CD being given, the greater of which EC exceeds a Quadrant, and the other CD is leſs than a Quadrant; if you divide by *Radius* the Sum of the Rectangles of the Sine of the one multiply'd by the Co-ſine of the other, the Quotient is the Sine of ED, the Difference of the Arcs CE and CD.

Sine of CE is EH $= b$, its Co-ſine is HB $= m$.
Sine of CD is CF $= c$, its Co-ſine is BF $= n$.
Sine of ED is GD $= a$.
I ſay, that $\frac{bn + cm}{r} = a$.

Demonſtration.

The $\triangle$s BCF and BIH are ſimilar; therefore

4. 6. *Eucl. El.* 1 $$n \,.\,.\, m :: \begin{cases} r \,.\,.\, \frac{rm}{n} = IB \\ c \,.\,.\, \frac{cm}{n} = HI \end{cases}$$

therefore 2 $$EI = b + \frac{cm}{n} = \frac{bn + mc}{n}$$

The $\triangle$s EBH and EIM are ſimilar.

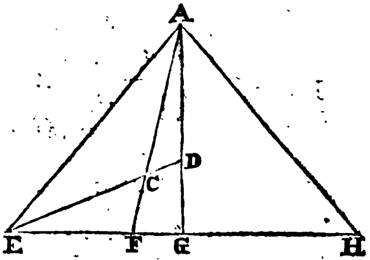
4. 6. *Eucl. El.* 3 $$r \,.\,.\, m :: \frac{bn + mc}{n} \,.\,.\, \frac{bnm + m^2c}{rn} = IM$$

The $\triangle$s IBM and DBG are ſimilar; wherefore

4. 6. *Eucl. El.*	4	$\frac{mr}{n} \,..\, \frac{bnm + m^2o}{rn} :: r \,..\, a$
Æq.	5	$a = \frac{bn + mc}{r}$. W. W. D.

PROBLEM XI.

Note, *This Problem was propos'd by then Capt.* Thornycraft *to, and ſolv'd by, the Reverend Mr.* Pierce Gould, *deceas'd; from whom I had it, and its Solution, and the four foregoing Lemma's.*



AH = AE, GE = GH, and EC = AC. AG = c is given.

AD = $\frac{2c}{3}$, and GD = $\frac{c}{3}$,

Sine of the ∠FAH = s, its Co-ſine = p, are given.

It is requir'd to find FG = y, by an Æquation, not exceeding an adfected Biquadratick.

Solution.

This Problem admits of three Caſes, for the ∠FAH given, is either leſs or greater than a right Angle, or elſe it is equal to a right Angle.

Caſe 1. Suppoſe the ∠ FAH given, to be leſs than a right Angle.

Suppoſe	1	AF = x, and FG = y
1 Ax. *Pl. Trig.*	2	$x \,..\, r :: y \,..\, \frac{yr}{x}$ = S ∠FAG
By the ſame	3	$x \,..\, r :: c \,..\, \frac{cr}{x}$ = S ∠AFG
By Lemma 2	4	$\frac{scr}{rx} - \frac{ypr}{rx} = \frac{sc - yp}{x}$ = S ∠GAH
By Lemma 3	5	$\frac{rys + crp}{rx} = \frac{ys + cp}{x}$ = S ∠GHA

By

By Lemma 2	6	$\frac{ccrs - cryp}{xxr} - \frac{yyrs - yrcp}{xxr} =$ $\frac{ccs - 2cyp - yys}{xx} = S\angle CAE = S\angle CEA$
Lemma 3	7	$\frac{yrsc - yyrp}{xxr} + \frac{crys + ccrp}{xxr} =$ $\frac{2ysc - yyp + ccp}{xx} = \text{Co-ſine} \angle CAE$
Lemma 2	8	$\frac{ys + cp}{x} \times \frac{2ysc - yyp + ccp}{rxx} -$ $\frac{sc - yp}{x} \times \frac{ccs - 2cyp - yys}{rxx} = S\angle DEG$ $= \frac{3yyssc - 2y^3ps + 6yccps - 3cp^2y^2}{rxxx}$ $+ c^3p^2 - c^3s^2$
2 Ax. *Pl. Trig.*	9	$\frac{ccs - 2cyp - yys}{xx} : \frac{2c}{3} :: \frac{sc - yp}{x} :$ $\frac{2sccx - 2cypx}{3c^2s - 6cyp - 3yys} = DE$
1 Ax. *Pl. Trig.*	10	$\frac{2sc^2x - 2cypx}{3ccs - 6cyp - 3y^2s} : r :: \frac{c}{3} :$ $\frac{rc^2s - 2rycp - ry^2s}{2scx - 2ypx} = S\angle DEG$
8 = 10	11	$\frac{3y^2s^2c - 2y^3ps + 6yc^2ps - 3cp^2y^2 + c^3p^2 - c^3s^2}{rx^3}$ $= \frac{rc^2s - 2rycp - ryys}{2scx - 2ypx}$
11 × rx^3	12	$3y^2s^2c - 2y^3ps + 6yc^2ps - 3cp^2y^2 + c^3p^2 - c^3s^2$ $= \frac{r^2c^2sx^2 - 2r^2x^2ycp - r^2y^2x^2s}{2sc - 2yp}$
47. 1. *Eucl. El.*	13	$yy + cc = xx$
therefore	14	$3y^2s^2c - 2y^3ps + 6yc^2ps - 3cp^2y^2 + c^3p^2 - c^3s^2$ $= \frac{r^2y^2c^2s + r^2c^4s - 2r^2y^3cp - 2r^2c^3yp - r^2y^4s - r^2y^2c^2s}{2cs - 2yp}$

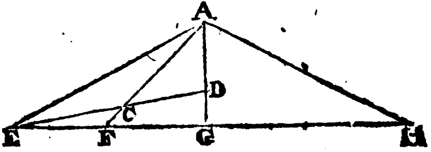
14 × $\overline{2sc-2yp}$ | 15 |
$$\left\{\begin{array}{l} 6y^2s^3c^2 - 4y^3pcs^2 + 12yc^3ps^2 - 6c^2p^2y^2s \\ + 2c^4p^2s - 2c^4s^3 + 4y^4p^2s + 6cp^3y^3 \\ - 2c^3p^3y - 6y^3pcs^2 + 2yc^3ps^2 - 12c^2p^2y^2s \\ = r^2c^4s - 2r^2y^3cp - 2r^2c^3yp - r^2y^4s \end{array}\right.$$

15 reduc'd | 16 |
$$\left\{\begin{array}{l} 4y^4p^2s - r^2y^4s + 6cp^3y^3 + 2r^2y^3cp - \\ 10y^3pcs^2 + 6y^2s^3c^2 - 18c^2p^2y^2s + 14yc^3ps^2 \\ + 2r^2c^3yp - 2c^3p^3y = r^2c^4s - 2c^4p^2s \\ - 2c^4s^3 \end{array}\right.$$

Æq. | 17 |
$$\left.\begin{array}{l} y^4 \\ + y^3 \times \dfrac{6cp^3 + 2r^2cp - 10pcs^2}{4p^2s + r^2s} \\ + y^2 \times \dfrac{6s^2c^2 - 18c^2p^2}{4pp + rr} \\ + y \times \dfrac{14c^3ps^2 + 2r^2c^3p - 2c^3p^3}{4p^2s + r^2s} \end{array}\right\} = \frac{r^2c^4 - 2c^4p^2 + 2c^4s^2}{4p^2 + r^2}$$

Caſe 2.

When ∠AFH is ⊏∟



Suppoſe as before | 1 | AF $= x$, and FG $= y$

then as before | 2 | $\dfrac{yr}{x} =$ S ∠FAG, and $\dfrac{cr}{x} =$ S ∠AFG

Lemma 4 | 3 | $\dfrac{scr}{rx} + \dfrac{pyr}{rx} = \dfrac{sc + py}{x} =$ S ∠GAH

Lemma 1 | 4 | $\dfrac{syr}{rx} - \dfrac{pcr}{rx} = \dfrac{sy - pc}{x} =$ S ∠GHA

Lemma 2 | 5 |
$$\left\{\begin{array}{l} \dfrac{src^2 + pycr}{rxx} - \dfrac{y^2rs + pycr}{rxx} = \\ \dfrac{c^2s + 2pyc - y^2s}{xx} = \text{S } \angle \text{EAC} \end{array}\right.$$

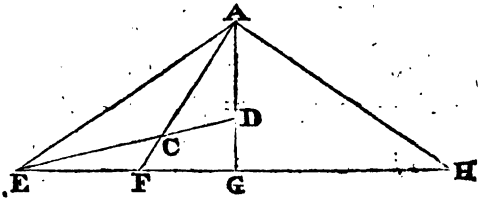
Lemma 3	6	$\frac{yrsc + y^2pr}{rxx} + \frac{crsy - c^2pr}{rxx} =$ $\frac{2syc + y^2p - c^2p}{xx} =$ Co-ſine $\angle$EAC
Lemma 2	7	$\frac{sy - pc}{x} \times \frac{2syc + y^2p - c^2p}{rxx} -$ $\frac{c^2s + 2pyc - y^2s}{rxx} \times \frac{sc + py}{x} =$ S $\angle$DEG $=$ $\frac{3s^2y^2c + 2sy^3p - 6syc^2p - 3p^2y^2c + c^3p^2}{rxxx}$ $\underline{- cccss}$
2 Ax. *Pl. Trig.*	8	$\frac{c^2s + 2pyc - y^2s}{xx} \,.\,.\, \frac{2c}{3} :: \frac{sc + py}{x} \,.\,.$ $\frac{2c^2sx + 2cpyx}{3c^2s + 6pyc - 3y^2s} =$ ED
1 Ax. *Pl. Trig.*	9	$\frac{2c^2sx + 2cpyx}{3c^2s + 6pyc - 3y^2s} \,.\,.\, r :: \frac{c}{3} \,.\,.$ $\frac{rc^2s - 2rpyc - ry^2s}{2csx + 2pyx} =$ S $\angle$DEG
$7 = 9$	10	$\frac{3s^2y^2c + 2sy^3p - 6syc^2p - 3p^2y^2c + c^3p^2 - c^3s^2}{rx^3}$ $= \frac{rc^2s - 2rpyc - ry^2s}{2csx + 2pyx}$
$10 \times rx^3$	11	$3s^2y^2c + 2sy^3p - 6syc^2p - 3p^2y^2c + c^3p^2$ $- c^3s^2 = \frac{r^2c^2x^2s + 2r^2pycx^2 - r^2x^2y^2s}{2cs + 2py}$
47. 1. *Eucl. El.*	12	$yy + cc = xx$
therefore	13	$3s^2y^2c + 2sy^3p - 6syc^2p - 3p^2y^2c + c^3p^2 - c^3s^2$ $= \frac{r^2c^4s + 2r^2py^3c + 2r^2pyc^3 - r^2y^4s}{2cs + 2py}$

13 ×

$13 \times \overline{2cs+2py}$	14	$\begin{cases} 6s^3y^2c^2 + 10s^2y^3pc - 14s^2yc^3p - 18p^2y^2c^2s \\ + 2c^4p^2s - 2c^4s^3 + 4p^2y^4s - 6p^3y^3c + 2c^3p^3y \\ = r^2c^4s + 2r^2py^3c + 2r^2pyc^3 - r^2y^4s \end{cases}$
14 reduc'd	15	$\left.\begin{cases} 4p^2y^4s + r^2y^4s + 10y^3s^2pc \\ -6y^3p^3c - 2y^3r^2pc + 6y^2s^3c^2 \\ -18y^2p^2c^2s + 2yc^3p^3 \\ -14ys^2c^3p - 2yr^2pc^3 \end{cases}\right\} \begin{cases} = r^2c^4s + 2c^4s^3 \\ - 2c^4p^2s \end{cases}$
Æq.	16	$\left\{\begin{array}{l} y^4 \\ + y^3 \times \dfrac{10s^2pc - 6p^3c - 2r^2pc}{4p^2s + r^2s} \\ + y^2 \times \dfrac{6s^2c^2 - 18p^2c^2}{4pp + rr} \\ + y \times \dfrac{2c^3p^3 - 14s^2c^3p - 2r^2pc^3}{4p^2s + r^2s} \end{array}\right\} = \dfrac{r^2c^4 + 2c^4s^2 - 2c^4p}{4pp + rr}$

Caſe 3. When $\angle FAG = \llcorner$

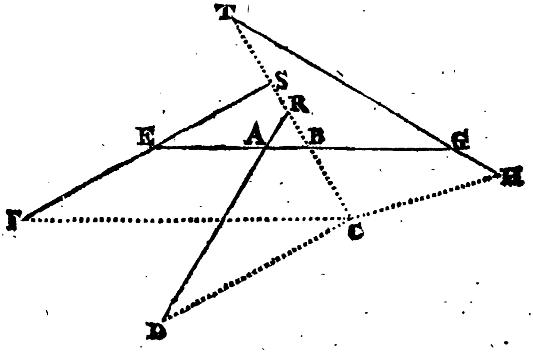
Then $\angle GAH = \angle AFG$, and $\angle GHA = \angle FAG$



Suppoſe as before	1 2	$AF = x$, and $FG = y$
then as before	3	$\begin{cases} S, \angle FAC = S, \angle GHA = \frac{yr}{x}, \text{ and} \\ S, \angle AFG = S, \angle GAH = \frac{rc}{x} \end{cases}$
Lemma 2	4	$\frac{rrcc}{rxx} - \frac{yyrr}{rxx} = \frac{rc^2 - ry^2}{xx} = S \angle EAC$
Lemma 3	5	$\frac{r^2cy}{rx^2} + \frac{r^2cy}{rx^2} = \frac{2rcy}{xx} = \text{Coſine } \angle EAC$
Lemma 2	6	$\begin{cases} \frac{yr}{x} \times \frac{2yrc}{rxx} - \frac{ccr - yyr}{rxx} \times \frac{rc}{x} = \frac{3yyrc - c^3r}{x^3} \\ = S \angle DEG \end{cases}$

By Ax. 2. *Pl. Trig.*	7	$\frac{ccr-yyr}{xx} \cdot\cdot \frac{2c}{3} :: \frac{rc}{x} \cdot\cdot \frac{2ccx}{3c^2-3y^2} =$ ED
Ax. 1. *Pl. Trig.*	8	$\frac{2ccx}{3c^2-3y^2} \cdot\cdot r :: \frac{c}{3} \cdot\cdot \frac{rc^2-ry^2}{2cx} =$ S ∟ DEG
$6=8$	9	$\frac{3yyrc-c^3r}{x^3} = \frac{rc^2-ry^2}{2cx}$
9 reduc'd	10	$3cy^2-c^3 = \frac{ccxx-yyxx}{2c}$
47. 1. *Eucl. El.*	11	$cc+yy=xx$
therefore	12	$3y^2c-c^3 = \frac{c^4-y^4}{2c}$
$12 \times 2c$	13	$6y^2c^2-2c^4 = c^4-y^4$
$3 + y^4 + 2c^4$	14	$y^4+6y^2c^2 = 3c^4$
Comp. □	15	$y^4+6y^2c^2+9c^4 = 9c^4+3c^4 = 12c^4$
15 ω 2	16	$y^2+3c^2 = \sqrt{12c^4}$
$16 - 3c^2$	17	$y^2 = \sqrt{12c^4}-3c^2$
17 ω 2	18	$y = \sqrt{\sqrt{12c^4}-3c^2}$

PROBLEM XII.



There are the Lines AB, AD, EF, GH, *&c.* given by Poſition, to find a Point from which if there be ſo many other

Lines drawn by Poſition, as CB, CD, CF, CH, *&c.* in the given ∠s CBA, CDA, CFE, CHG, *&c.* the Product of the Multiplication, of certain of theſe Lines may be equal to the Product of the Multiplication of the reſt; or, which is the ſame Thing, that one may be to another in a given Reaſon.

I will ſolve this Problem, as tho' there were the 4 *Lines AB, AD, EF, and GH, and no more given, and ſo many more requir'd to be drawn: The Manner of doing which, will (I preſume) give one an Inſight in ſolving the propos'd Problem, in reſpect of any determin'd Number of ſuch Lines.*

1ſt, I conſider the Thing as done; and, to avoid Confuſion, I conſider one of the given Lines, and one of the requir'd, as AB and CB, as the Principal to which I endeavour to refer and compare all the reſt.

Then I produce all the Lines, ſo as to croſs theſe two, if poſſible, (as I ſuppoſe it is) to wit, AB in the Points A, E, and G, and BC in R, S, and T. Then I put

$$\left\{\begin{array}{l} AB = b \\ AE = c \\ AG = d \\ \text{Then, } BG = d - b \\ EB = c + b \\ BC = y = z \end{array}\right. \quad \left\{\begin{array}{l} \text{In this} \\ \text{Fig.} \end{array}\right.$$

2dly, Becauſe in the △ ABR the (a) ∠s A and B and (b) conſequently R are given; the Ratio of the Sides AB and BR is therefore given, which I put as f to g; then .... $f \,..\, g :: b \,..\, \frac{bg}{f} = BR$.

(a) *Conſtr.*

(b) 32. 1. *Eucl. El.*

(c) *By next preceding Step.*

3. Conſequently .. $y + \frac{bg}{f} = CR$.

4. Likewiſe in the △ DRC the ∠s (c) R (a) D and (b) C being given, the Ratio of the Sides CR and CD are given; which I put as f to h,

therefore .... $f \,..\, h :: y + \frac{bg}{f} \,..\, \frac{hy}{f} + \frac{hbg}{ff} = CD$.

5. Again, in the △ ESB the (a) ∠s B, E, and (b) S being given, the Ratio of the Sides BE and BS is given, which I put as f to k; therefore $f \,..\, k :: c + b \,..\, \frac{kc + kb}{f} = BS$.

6. And conſequently $y + \frac{kc + kb}{f} = CS = \frac{fy + kc + kb}{f}$.

7. Again, in the △ FSC the ∠s (a) F, (c) S, and (b) C being given, the Ratio of the Sides CS and CF is therefore given, which I put as f to l; then

$$f \,..\, l :: \frac{fy + kc + kb}{f} \,..\, \frac{lfy + lkc + lkb}{ff} = CF.$$

8. Again, in the △ BGT the ∠s (a) B, G, and (b) T being given, the Ratio of BG to BT is given, which I put as f to m, then $f \,..\, m :: d - b \,..\, \frac{md - mb}{f} = BT$.

9. Conſequently $y + \frac{md - mb}{f} = TC = \frac{fy + md - mb}{f}$.

10. Laſtly, in the △ TCH the ∠s (a) H, (c) T, and (b) C, being given, the Ratio of CT to CH is given, which I put as f to n; then $f \,..\, n :: \frac{fy + md - mb}{f} \,..\, \frac{nfy + nmd - nmb}{ff} = CH$.

11. According to the Import of the Problem CB × CF = CD × CH; that is, from the foregoing Steps,

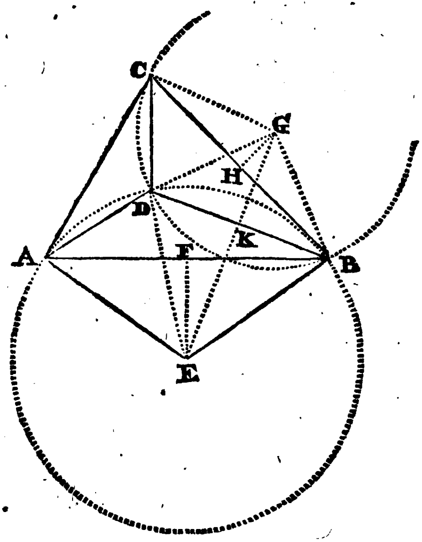
$$y \times \frac{lfy + lkc + lkb}{ff} = \frac{fhy + hbg}{ff} \times \frac{nfy + nmd - nmb}{ff}$$

12. The 11th being reduc'd, gives

$$yy = \frac{\begin{array}{l} -lf^2kc \\ -lf^2kb \\ +fhnmd \\ -fhnmb \\ +fhbgn \end{array} y \quad \begin{array}{l} \\ +hbgnmd \\ \\ -hb^2gnm \\ \\ \end{array}}{lf^3 - f^2hn}$$

PRO-

PROBLEM XIII.



AB = 20000
AC = 15000 } Given.
BC = 18000

EB = 12521 . 286 +
GB = 9317 . 445 +
DB = 13888 . 610 +
DA = 8283 . 832 +
DC = 8406 . 944 +

∠ADB = 127° 00′
∠BDC = 105 : 00 } Given.
∠CDA = 128 : 00

∠FEB = 53 : 00
∠EBA = 37 : 00
∠HGB = 75 : 00
∠GBC = 15 : 00
∠CBA = 46 : 08
∠GBE = 98 : 08
∠GEB = 33 : 41
∠DBA = 19 : 19
∠DBC = 26 : 49

The Diſtances AB, AC, and BC between 3 Towers A, B, and C, not ſtanding in ſtraight Line being given ſeverally; alſo a 4th Tower being ſuppos'd to ſtand within the △ ABC, as at D, and the Meaſures of the ∠s ADB, BDC, and CDA being given ſeverally; to find the Diſtances between the 4th Tower D, and each of the other 3, *viz.* the Meaſure of the Lines DA, DB, and DC.

Conſtruction.

On AB make an Iſoſceles △ AEB, whoſe ∠ AEB, or its Complement to 360°, ſhall be = 2 ∠s ADB; on E, as a Center, deſcribe the Circle ADB paſſing thro' A and B.

On BC make an Iſoſceles △ BGC, whoſe ∠ BGC, or its Complement to 360°, ſhall be = 2 ∠s CDB: On G, as a Center, deſcribe the Circle BDC paſſing thro' B and C, and cutting the former Circle in D, draw AD, DB, and DC, and the Conſtruction is perform'd.

Method of Calculation, by John Kerſey.

On AB let fall the ⊥ EF, and on CB the ⊥ GH. Draw DE, DG, and EG.

1. Subtract the given ∠ ADB from two right Angles (*viz.* from 180°) the Remainder ſhall be the Sum of the unknown ∠s DAB, and DBA, by 32. 1. *Eucl. El.*

2. For as much as (by 20. 3. *Eucl. El.*) ∠ DEB = 2 ∠s DAB, and ∠ DEA = 2 ∠s DBA; It follows, that ∠ AEB = 2 ∠s DAB + 2 ∠s DBA; therefore, in the △ FEB right-angled at F, the ∠ FEB (that is ½ ∠ AEB) = ∠ DAB + ∠ DBA is given; and by Suppoſition FB = ½ AB is given; therefore the Semidiameter EB = ED = EA ſhall be given alſo.

3. By arguing as before in 1ſt and 2d Steps, ∠ HGB = ∠ HGC is given; alſo GD = GC = GB, the Semidiameter of the Circle GBDC is given.

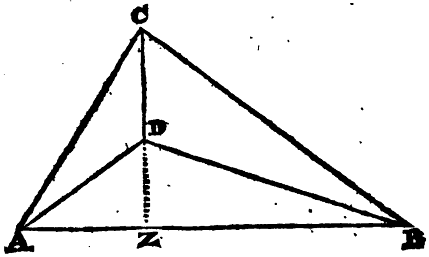
4. Becauſe EF ⊥ AB, and ∠ FEB is given as before, therefore ∠ EBA the Complement of ∠ FEB to a right Angle is given: Likewiſe ∠ GBC the Complement of ∠ HGB to a right Angle is given, and the ∠ CBA is given, for it may be found out by the three given Sides AB, AC, and BC; therefore ∠ GBE the Sum of thoſe three Angles EBA, CBA, and GBC is given.

5. In the △ GBE the Sides GB and EB (to-wit the Semidiameters of the two Circles GBDC, and EADB) are given ſeverally, as alſo the ∠ GBE comprehended by thoſe Sides; therefore, the ∠ GEB is given alſo.

6. Becauſe the two △s EGB and EGD have two Sides GB and EB equal to the two Sides GD, ED, *viz.* GB = GD, and EB = ED, alſo the Baſe GE common to both thoſe △s, the ∠s contain'd under equal right Lines ſhall be equal, *viz.* ∠GEB = ∠GED = $\frac{1}{2}$ ∠DEB; but ∠GED (= $\frac{1}{2}$ ∠DEB) is given in the 5th Step, and (*per* 20. 3. *Eucl. El.*) ∠DAB is = $\frac{1}{2}$ ∠DEB (= ∠GEB); therefore ∠DAB is given.

Now in the △ADB there is given ∠DAB, as alſo ∠ADB, and the Side AB; therefore the Sides DB and DA (to-wit two of the Diſtances ſought) are given alſo.

7. And laſtly, in the △DAC, there is given DA, as alſo AC, and ∠ADC; therefore the 3d Diſtance ſought is given.



An Algebraical Solution of Prob. 13.

Suppoſe the ∠s ADB, CDB, and ADC to be obtuſe, and the ∠s CAB and ABC acute.

AB = b
BC = c
CA = d
S ∠ADB = f; its Co-ſine = p Given.
S ∠CDB = g . . . .
S ∠ADC = h . . . .
S ∠CAB = l; its Co-ſine = k
S ∠ABC = m; its Co-ſine = n

Requir'd to find AD = a = ?

Draw DZ $\perp$ AB.

Solution.

2 Ax. *Pl. Trig.* — 1 $b\,..\,f::a\,..\,\frac{fa}{b}=S\angle DBA$

2 $\sqrt{rr-\frac{ffaa}{bb}}=\Sigma\angle DBA=S\angle BDZ$

Lemma 4. — 3 $\frac{ffa}{rb}+\frac{p}{r}\sqrt{rr-\frac{ffaa}{bb}}=S\angle ADZ$

Lemma 1. — 4 $\frac{f}{r}\sqrt{rr-\frac{ffaa}{bb}}-\frac{pfa}{rb}=S\angle DAZ$

Lemma 2. — 5 $\begin{cases}\frac{lffa}{rrb}+\frac{lp}{rr}\sqrt{rr-\frac{ffaa}{bb}}-\\ \frac{kf}{rr}\sqrt{rr-\frac{ffaa}{bb}}+\frac{kpfa}{rrb}=S\angle DAC\end{cases}$

Lemma 2. — 6 $\frac{m}{r}\sqrt{rr-\frac{ffaa}{bb}}-\frac{nfa}{rb}=S\angle DBC$

to Problem XI.

2 Ax. *Pl. Trig.* — 7 $\begin{cases}g\,..\,c::\frac{m}{r}\sqrt{rr-\frac{ffaa}{bb}}-\frac{nfa}{rb}\,..\\ \frac{cm}{gr}\sqrt{rr-\frac{ffaa}{bb}}-\frac{cnfa}{grb}=CD\end{cases}$

2 Ax. *Pl. Trig.* — 8 $\begin{cases}d\,..\,b::\frac{cm}{gr}\sqrt{rr-\frac{ffaa}{bb}}-\frac{cnfa}{grb}\\ ..\,\frac{bcm}{dgr}\sqrt{rr-\frac{ffaa}{bb}}-\frac{bcnfa}{dgbr}\\ =S\angle CAD=(\text{per }5\text{th})\,\frac{lffa}{rrb}+\\ \frac{lp}{rr}\sqrt{rr-\frac{ffaa}{bb}}-\frac{kf}{rr}\sqrt{rr-\frac{ffaa}{bb}}\\ +\frac{kpfa}{rrb}\end{cases}$

8 × r and Tranſp. | 9 | $$\overline{\frac{bcm}{dg}-\frac{lp}{r}+\frac{kf}{r}}\times\sqrt{rr-\frac{ffaa}{bb}}=\overline{\frac{bcnf}{dgb}+\frac{lff}{rb}+\frac{kpf}{rb}}\times a$$

Now for $\frac{bcm}{dg}-\frac{lp-kf}{r}$ put t

And for $\frac{bcnf}{dgb}+\frac{lff+kpf}{rb}$ put w;

then the 9th Æquation will become

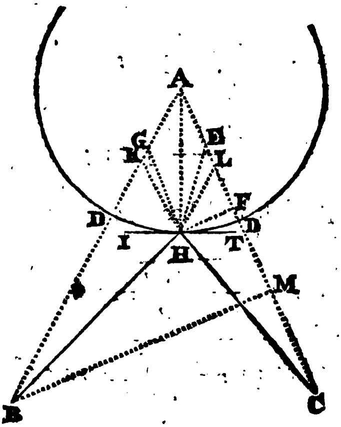
| 10 | $$t\sqrt{rr-\frac{ffaa}{bb}}=wa$$

10 ◎ 2 | 11 | $$tt\times\overline{rr-\frac{ffaa}{bb}}=wwaa$$

11 reduc'd | 12 | $$a=\frac{trb}{\sqrt{ttff+bbww}}$$

PROBLEM XIV.

Which is ſaid to have buſy'd the ableſt Mathematicians in *Europe*.



There being given upon a Plane two Points B and C, and the Circle DHD, whoſe Center is A, and the Radius AD; to find upon the Circumference the Point H, thro' which drawing to the two Points B and C given the right Lines BH, CH, and the Tangent $IT \perp AH$, the two Angles BHI, CHT ſhall be equal.

Which is the ſame Thing as to require the Point of Reflexion H upon the convex or concave Surface of a ſpherical Glaſs given, the Eye being plac'd in B, and the Object in C.

Solution by Mr. L'Abbè de Catalan.

Having joyn'd AB, AC, and AH, draw HF, and BM $\perp$s AC; draw $HK \parallel AC$, and $HL \parallel AB$; draw HE ſo that $\angle HEA = \angle AHC = \angle AHB$, and the Line HG that $\angle HGA = \angle AHB$. This Preparation made, we conſider that $\triangle$s AHB, AHG are alike; alſo that $\triangle$s AHC, AHE are alike; alſo that $\triangle$s GHK, EHL are alike. This done,

Suppoſe AB $= a$
AC $= b$

$$\left.\begin{array}{l} AM = c \\ BM = d \end{array}\right\}$$

For in △ ABM the ∟AMB the ∠MAB and Side BA are given, and conſequently AM and MB.

AD = AH $= r$

$$\left.\begin{array}{l} AF = x \\ HF = y \end{array}\right\}$$

then

In the like △s ABM, LHF we have HL $= \frac{ay}{d}$, and FL $= \frac{cy}{d}$, which taken from AF $= x$, we have AL $= x - \frac{cy}{d}$ = HK, and AK = HL $= \frac{ay}{d}$.

In the like △s AHB, AHG, we have AG $= \frac{rr}{a}$; and in the like △s AHC and AEH, we have AE $= \frac{rr}{b}$, which being ſubtracted from AL $= x - \frac{cy}{d}$ we have EL $= x - \frac{cy}{d} - \frac{rr}{b}$, alſo GK $= \frac{ay}{d} - \frac{rr}{a}$.

In the like △s GHK, EHL

HK . . GK : : HL . . EL

$$x - \frac{cy}{d} \,..\, \frac{ay}{d} - \frac{rr}{a} :: \frac{ay}{d} \,..\, x - \frac{cy}{d} - \frac{rr}{b}.$$

Hence $xx - \frac{2cyx}{d} - \frac{rrx}{b} + \frac{ccyy}{dd} + \frac{rrcy}{db} = \frac{aayy}{dd} - \frac{rry}{d}$

or $xx - \frac{2cxy}{d} - \frac{rrx}{b} = \frac{aayy - ccyy}{dd} - \frac{crry}{bd} - \frac{rry}{d}$.

And if for $aa - cc$ we put dd its equal, becauſe of the ∟d △ AMB, we ſhall have this Æquation,

$$xx - \frac{2cxy}{d} - \frac{rrx}{b} = yy - \frac{crry}{db} - \frac{rry}{d}.$$

Add yy to each Part of the next foregoing Step, and you'll have $\overline{xx+yy}\,(rr) - \frac{2cxy}{d} - \frac{rrx}{b} = 2yy - \frac{crry}{db} - \frac{rry}{d}$.

Subſtitute $\sqrt{rr-xx}$ for y, and $rr - xx$ for yy;

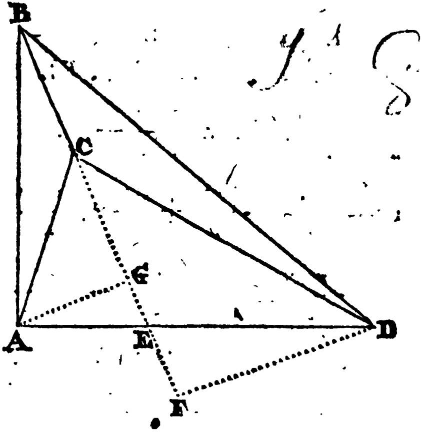
Then $rr - \frac{2cx}{d}\sqrt{rr-xx} - \frac{rrx}{b} = 2rr - 2xx - \frac{crr}{db}\sqrt{rr-xx} - \frac{rr}{d}\sqrt{rr-xx}$. Tranſp. and abb.

$$2xx - \frac{rrx}{b} - rr = \overline{\frac{2cx}{d} - \frac{crr}{db} - \frac{rr}{d}} \times \sqrt{rr-xx}.$$

Now by ſquaring each Part, &c. you'll gain an Æquation, wherein one of the Values of x is $=$ AF.

PROBLEM XV.

Which is the ſame with Problem 14, *and thus ſolv'd by the Reverend Mr.* Gould.



Triangle ABD, and the Length of the Line BC given.

Requir'd to draw AC and CD, ſo that the ∠ACE = ∠ECD.

Sine ∠ABD $= s$, its Co-ſine $= p$, AB $= b$, BC $= c$, BD $= d$, } Given.

FD $= x$ ſought.

Produce BC to F, and from the Points D and A let fall DF and AG both Perpendiculars to BF; and ſuppoſe BF $= y$, then $yy = dd - xx$.

Solution.

	1	$CF = y - c$
Ax. 1. *Pl. Trig.*	2	$d \,..\, r :: x \,..\, \frac{xr}{d} = S\angle FBD$
	3	$d \,..\, r :: y \,..\, \frac{yr}{d} = S\angle FDB = \Sigma\angle FBD$
Lemma 2. to Prob. 11.	4	$\frac{syr - pxr}{rd} = \frac{sy - px}{d} = S\angle ABG$
Lemma 3. to ſaid Prob. 11.	5	$\frac{sxr + pyr}{rd} = \frac{sx + py}{d} = S\angle BAG$
1 Ax. *Pl. Trig.*	6	$r \,..\, b :: \frac{sy - px}{d} \,..\, \frac{syb - pxb}{rd} = AG$
	7	$r \,..\, b :: \frac{sx + py}{d} \,..\, \frac{sxb + pyb}{rd} = BG$
	8	$CG = BG - BC = \frac{sxb + pyb}{rd} - c$
The Δs ACG and DCF are alike.	9	$\frac{syb - pxb}{rd} \,..\, \frac{sxb + pyb - rdc}{rd} :: x \,..\, y - c$
Æq.	10	$\left\{\begin{array}{l} syyb - pxby - sybc + pxbc = \\ \quad sxxb + pyxb - rdcx \end{array}\right.$

10 reduc'd (after ſubſtituting $\overline{dd-xx}|^{\frac{1}{2}}$ for y) | 11 |

$$\left\{\begin{array}{l}\overline{4s^2b^2+4p^2b^2}\times x^4\\ \overline{+4pb^2sc-4sb^2pc-4sbrdc}\times x^3\\ \left.\begin{array}{l}\overline{+p^2b^2c^2+2rdc^2pb-4d^2s^2b^2}\\ \overline{+r^2d^2c^2-4p^2b^2d^2+s^2b^2c^2}\end{array}\right\}\times x^2\\ \overline{+2d^2sb^2pc+2d^3sbrc-4pb^2scd^2}\times x\end{array}\right\}=\left\{+s^2b^2d^2c^2-d^4s^2b^2\right.$$

N: B. Part III. begins Page 417, Signature H h h.

PART

PART III.

Of Exhauſtions.

THE Method of *Exhauſtions* is founded upon the following

Lemma I.

If by adding a Quantity, be it ever ſo little, that is, a Quantity indefinitely little, to either of two given Quantities, the Sum thereby becomes greater than the other given Quantity; I ſay, the ſaid two given Quantities muſt be equal.

Expoſition.

If b and c be $=$ the two given Quantities finite, and a $=$ an indefinitely little Quantity:

And if $b + a \sqsubset c$,
and $c + d \sqsubset b$; I ſay that $b = c$.

Demonſtration.

For if you ſuppoſe that either of them as b could be $\sqsubset$ the other c; it muſt be by ſome Quantity, which ſuppoſe $= d$, then b would be $= c + d$; And the Conſequence would be that $d \sqsupset a$.

But altho' d had been leſs than

$$b \times \frac{1}{999999 \text{ \&c. till you have \textit{Octillions} of 9's writ one after another}},$$

Or leſs than $b \times$, the Square of that Number; or leſs than $b \times$ by its Cube, &c. Yet $b \times$ by ſome Power of the ſaid Number muſt be $\sqsupset d$ (by *Poſt. Lib.* 10. *Eucl. El.*) which Power $\times\, b$ let a (for it is evident that it may) be equal to, or leſs than; then a will be $\sqsupset d$, which is contrary to the foregoing Conſequence; and therefore by ſuppoſing that b is $\sqsubset c$, a Contradiction enſues, which proves that b is not $\sqsubset c$; and for the ſame Reaſons c is not $\sqsubset b$, conſequently b is $= c$. Q. D. E.

That part of *Exhauſtions* which treats of Quadratures is ~~likewiſe founded upon the following *Lemma*; that~~ is, upon its *Scholia* and *Coroll.*

~~*Lemma*~~ II.

If n be $=$ an indefinite Number, and $p =$ ſuch a Number as that the Sum (s) of $1^p, 2^p, 3^p, 4^p$, &c. continued to n Terms be $= \frac{n^{p+1}}{p+1}\phi$, Suppoſe $= \frac{n^{p+1}}{p+1} + a \times n^p \phi$: Then I ſay the Sum ($z$) of $1^{p+1}, 2^{p+1}, 3^{p+1}, 4^{p+1}$, &c. continu'd to n Terms, will be $= \frac{n^{p+2} + \overline{pa+1} \times n^{p+1}}{p+2}\phi$.

Demonſtration.

It is evident that z is $=$

$$\left\{\underbrace{\begin{array}{l} 1^p + 2^p + 3^p + \&c. + n^p \\ 1^p + 2^p + 3^p + \&c. + n^p \\ 1^p + 2^p + 3^p + \&c. + n^p \\ 1^p + 2^p + 3^p + \&c. + n^p \end{array}}_{\text{\&c. to } n \text{ Ranks continu'd.}}\right\} - \left\{\underbrace{\begin{array}{l} 0 \\ 1^p \\ 1^p + 2^p \\ 1^p + 2^p + 3^p \end{array}}_{\text{\&c. to } n \text{ Ranks continu'd.}}\right\} = \left\{\underbrace{\begin{array}{r} 1^p + 2^p + 3^p + \&c. + n^p \\ 2^p + 3^p + \&c. + n^p \\ 3^p + \&c. + n^p \\ + \&c. + n^p \end{array}}_{\text{\&c. to } n \text{ Ranks continu'd.}}\right\}$$

Now if s be $= \frac{n^{p+1}}{p+1} + a \times n^{p}\phi$; then the Sum of the foregoing greater Series is $= \frac{n^{p+2}}{p+1} + a \times n^{p+1}\phi$.

Again, in order to find the Sum of the leſſer Series; conſider, that ſince o is the firſt Rank thereof; 1^p the 2d, $1^p + 2^p$ the 3d, &c. that $1^p + 2^p + 3^p + 4^p +$ &c. $+ \overline{n-1}|^p$ will be = the nth Rank; alſo $1^p + 2^p + 3^p + 4^p +$ &c. $+ \overline{n-2}|^p =$ the $\overline{n-1}$*th* Rank; alſo $1^p + 2^p + 3^p + 4^p +$ &c. $+ \overline{n-3}|^p =$ the $\overline{n-2}$th Rank, *&c.* of the ſaid leſſer Series: Wherefore if s be $= \frac{n^{p+1}}{p+1} + a \times n^p \phi$, the Sum of the ſaid nth Rank will be $= \frac{\overline{n-1}|^{p+1}}{p+1} + a \times \overline{n-1}|^p \phi$; alſo the Sum of the ſaid $\overline{n-1}$*th* Rank will be $= \frac{\overline{n-2}|^{p+1}}{p+1} + a \times \overline{n-2}|^p \phi$; alſo the Sum of the ſaid $\overline{n-2}$*th* Rank $= \frac{\overline{n-3}|^{p+1}}{p+1} + a \times \overline{n-3}|^p \phi$, &c. Wherefore the Sum of the ſaid leſſer Series is $= \frac{\overline{n-1}|^{p+1} + \overline{n-2}|^{p+1} + \overline{n-3}|^{p+1} + \text{\&c. to } \overline{n-1}}{p+1}$ Terms continu'd; $+ a \times \overline{\overline{n-1}|^p + \overline{n-2}|^p + \overline{n-3}|^p + \text{\&c. to } n-1 \text{ Terms continu'd}}$: But $\frac{\overline{n-1}|^{p+1} + \overline{n-2}|^{p+1} + \overline{n-3}|^{p+1} + \text{\&c.}}{p+1}$ is manifeſtly $= \frac{z - n^{p+1}}{p+1}$: And if $a \times \overline{\overline{n-1}|^p + \overline{n-2}|^p + \overline{n-3}|^p + \text{\&c.}}$ be $= a \times \frac{\overline{n-1}|^{p+1}}{p+1}\phi$; then the Difference of the Greater and Leſſer foregoing Series is

=

$= \frac{n^{p+2}}{p+1} + a \times n^{p+1}\phi + \frac{z - n^{p+1}}{p+1} - a \times \frac{\overline{n-1}|^{p+1}}{p+1}\phi$, which by Sup. is $= z = \frac{n^{p+2} + \overline{pa+a} \times n^{p+1} - z + n^{p+1} - a \times n^{p+1}\phi}{p+1}$, (becauſe $\overline{n-1}|^{p+1}$ is $= n^{p+1}\phi$;) therefore $n^{p+2} + n^{p+1} + pa \times n^{p+1}\phi - z = pz + z$; conſeq. $\frac{n^{p+2} + \overline{pa+1} \times n^{p+1}}{p+2}\phi = z$.

Scholium I.

If n be $=$ an indefinite Number, and p be $=$ 1, then $1^p + 2^p + 3^p +$ &c. $+ n^p$ is (by what has been ſaid in ÷) $= \frac{n^{p+1}}{p+1} + \frac{1}{2}n^p$; conſequently $1^{p+1} + 2^{p+1} + 3^{p+1} +$ &c. $+ n^{p+1}$ (will be by our *Lem.* 2.) $= \frac{n^{p+2}}{p+2} + \frac{1}{2}n^{p+1}\phi$ ($= \frac{n^{p+2}}{p+2} + \frac{\frac{1}{2}p+1}{p+2} \times n^{p+1}\phi$.) Again, if p be $= 2$, then $1^p + 2^p + 3^p +$ &c. $+ n^p$ (by this *Scholium*) is $= \frac{n^{p+1}}{p+1} + \frac{1}{2}n^p\phi$; therefore $1^{p+1} + 2^{p+1} + 3^{p+1} +$ &c. $+ n^{p+1}$ will be (by our *Lem.* 2.) $= \frac{n^{p+2}}{p+2} + \frac{1}{2}n^{p+1}\phi$, &c.

Corollary I.

Wherefore if n be $=$ an indefinite Number, and $p =$ any Affirmative whole Number $\sqsubset 1$, $1^p + 2^p + 3^p +$ &c. $+ n^p$ will be $= \frac{n^{p+1}}{p+1} + \frac{1}{2}n^p\phi$:

And conſequently $1^p + 2^p + 3^p +$ &c. $+ \overline{n-1}|^p = \frac{n^{p+1}}{p+1} + \frac{1}{2}n^p\phi - n^p = \frac{n^{p+1}}{p+1} - \frac{1}{2}n^p\phi$.

Scholium II.

Scholium II.

1. Again, by *Scholium* to *Prob.* 2. *Chap.* 3. of this part 3. it is further demonſtrated, that p being $=$ any affirmative whole Number $1^{\frac{1}{p}} + 2^{\frac{1}{p}} + 3^{\frac{1}{p}} + \&c. + n^{\frac{1}{p}}$ is $= \frac{n^{\frac{1}{p}+1}}{\frac{1}{p}+1} \theta$; therefore $1^{\frac{1}{p}+1} + 2^{\frac{1}{p}+1} + 3^{\frac{1}{p}+1} + \&c. + n^{\frac{1}{p}+1}$ is (by our *Lemma* 2.) $= \frac{n^{\frac{1}{p}+2}}{\frac{1}{p}+2} \theta$; &c.

2. Likewiſe by the ſaid *Scholium* to *Prob.* 2. *Chap.* 3. of this Part 3. it is demonſtrated that $1^{\frac{1}{p}} + 2^{\frac{1}{p}} + 3^{\frac{1}{p}} + \&c.$ continu'd to $n - 1$ Terms is $= \frac{n^{\frac{1}{p}+1}}{\frac{1}{p}+1} \theta$: Let us ſuppoſe it $= \frac{n^{\frac{1}{p}+1}}{\frac{1}{p}+1} - x \times n^{\frac{1}{p}} \phi$; then the Sum of this Series continu'd to n Terms is $= \frac{n^{\frac{1}{p}+1}}{\frac{1}{p}+1} + \overline{1 - x} \times n^{\frac{1}{p}} \phi$: And $1^{\frac{1}{p}+1} + 2^{\frac{1}{p}+1} + 3^{\frac{1}{p}+1} + \&c.$ continu'd to n Terms, will be (by our *Lemma* 2.) $= \frac{n^{\frac{1}{p}+2} + \overline{\frac{1-x}{p} + 1} \times n^{\frac{1}{p}+1} \phi}{\frac{1}{p}+2}$; conſequently the Sum of the laſt Series continu'd only to $\overline{n - 1}$ Terms is $= \frac{n^{\frac{1}{p}+2} - \overline{\frac{1}{p} x + 1} \times n^{\frac{1}{p}+1} \phi}{\frac{1}{p}+2} = \frac{n^{\frac{1}{p}+2}}{\frac{1}{p}+2} \theta$.

From what has been ſaid it is manifeſt, that q being likewiſe $=$ any affirmative whole Number

$1^{\frac{1}{p}+q} + 2^{\frac{1}{p}+q} + 3^{\frac{1}{p}+q} + \&c. = 1^{\frac{1+pq}{p}} + 2^{\frac{1+pq}{p}} + 3^{\frac{1+pq}{p}} + \&c.$

continu'd to $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\}$ Terms is $= \dfrac{n^{\frac{1+pq}{p}+1}}{\frac{1+pq}{p}+1} \left\{ \begin{matrix} \ominus \\ \ominus \end{matrix} \right\}$.

Wherefore it may be demonſtrated by the like Method with that us'd in *Prob.* 2. and its *Sch.* in *Chap.* 3. of this Part 3. that $1^{\frac{p}{1+pq}} + 2^{\frac{p}{1+pq}} + 3^{\frac{p}{1+pq}} +$ &c. continu'd to $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\}$ Terms is $= \dfrac{n^{\frac{p}{1+pq}+1}}{\frac{p}{1+pq}+1} \left\{ \begin{matrix} \ominus \\ \ominus \end{matrix} \right\}$.

Now, tho' p and q be limited ſo as to be equal to whole Numbers, yet there is no Number either whole or fracted but $\dfrac{p}{1+pq}$, or $\dfrac{1+pq}{p}$ may be equal to; therefore

Corollary II.

If m be $=$ any affirmative Number whatſoever, and $n =$ an Indefinite Number $1^m + 2^m + 3^m + 4^m +$ &c. continu'd to n and $\overline{n-1}$ Terms, will be $= \dfrac{n^{m+1}}{m+1} \ominus$, and $\dfrac{n^{m+1}}{m+1} \ominus$ reſpectively.

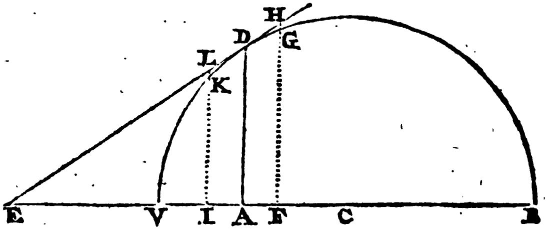
CHAP.

CHAP. I.

How to draw Tangents to Curve Figures.

PROP. I.

To draw a Tangent to a Circle.



$$\left.\begin{array}{l} VB = 2r \\ VA = x;\ \text{then } AB = 2r - x,\ \text{and} \\ \text{the Property of a Circle} \end{array}\right\}\text{Given.}$$

Required to find $AE = t$. *Note,* $AD \perp VB$.

Preparation.

Produce the Tangent ED to H, and imagine the Line HGF to be ||, and indefinitely-near DA, and LKI || and indefinitely-near DA, and (if you will) let AI be = AF = a.

Solution.

By the Property of a Circle AB × VA = □ DA; that is, $\overline{2r - x} \times x = 2rx - xx = DAq$: and FB × VF = □FG; that is, $\overline{2r - x - a} \times \overline{x + a} = 2rx + 2ra - x^2 - 2ax - aa = FGq$:

The $\triangle^s$ EHF and EDA are ſimilar; wherefore
$EA \cdot\cdot EF :: AD \cdot\cdot FH$; and
$EAq : \cdot\cdot EFq : :: ADq : FHq$: That is,
$tt \cdot\cdot tt + 2ta + aa :: 2rx - xx \cdot\cdot FHq$:

$$= \frac{2rxtt + 4rxta + 2rxaa - xxtt - 2txxa - aaxx}{tt};$$

But FH $\sqsubset$ FG (let a be ever ſo ſmall, unleſs it be = to, or leſs than nothing, as it is not by Suppoſition;) therefore $FHq : \sqsubset FGq$;

that is, $\frac{2rxtt + 4rxta + 2rxaa - xxtt - 2txxa - aaxx}{tt}$ $\sqsubset 2rx + 2ra - xx - 2ax - aa$.

The foregoing Step × tt, produces $2rxtt + 4rxta + 2rxaa - xxtt - 2txxa - aaxx \sqsubset 2rxtt + 2ratt - xxtt - 2axtt - aatt$: Therefore $4rxta + 2rxaa - 2txxa - a^2x^2 \sqsubset 2ttra - 2ttax - ttaa$: And by dividing each part by a, and tranſp. $4rxt + 2rxa + tta - x^2a \sqsubset 2ttr - 2ttx + 2txx$: But $2rxa \sqsubset xxa$, and conſequently $2rxa + tta$ muſt be $\sqsubset x^2a$; Wherefore the foregoing Step may be deſign'd thus, $4rxt \sqsubset 2ttr - 2ttx + 2txx$.

2*dly*, $IB \times VI = KIq$, that is $\overline{2r - x + a} \times \overline{x - a} = 2rx - 2ra - xx + 2ax - aa = KIq$.

The $\triangle^s$ EIL and EAD are Similar; wherefore $EA \cdot\cdot AD :: EI \cdot\cdot IL$, that is, $t \cdot\cdot \overline{2rx - xx}|^{\frac{1}{2}} :: t - a \cdot\cdot IL$. Conſequently $tt \cdot\cdot 2rx - xx :: tt - 2ta + aa \cdot\cdot ILq$:

$$= \frac{2rxt^2 - 4rxta + 2rxa^2 - x^2t^2 + 2tx^2a - x^2aa}{tt}$$

But $ILq : \sqsubset IKq$; that is

$$\frac{2rxtt - 4rxta + 2rxa^2 - x^2t^2 + 2tx^2a - x^2a^2}{tt} \sqsubset$$

$2rx - 2ra - xx + 2xa - aa$; wherefore $2rxtt - 4rxta + 2rxaa - xxt^2 + 2tx^2a - a^2x^2 \sqsubset 2rxtt - 2ratt - x^2t^2 + 2axtt - aatt$; conſequently $2txx + 2rtt - 2ttx + 2rxa + tta - x^2a \sqsubset 4rxt$; that is $2tx^2 + 2rt^2 - 2t^2x \sqsubset 4rxt$. That is, in Words, any Quantity whatſoever, that is more than nothing, being added to either of the Quantities $4rxt$, or $2txx + 2ttr - 2ttx$, makes the Sum greater than the other Quantity; conſequently by * our *Lemma* 1. $4rxt = 2txx + 2ttr - 2ttx$; whence $4rx - 2xx = 2tr - 2tx$: Finally, by dividing each Part by $2r - 2x$ you have $\frac{4rx - 2xx}{2r - 2x}$ $= t = \frac{2rx - xx}{r - x}$.

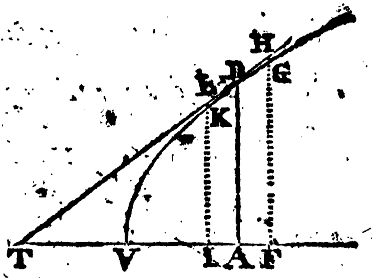
*Lemma 1. to Exhauſtions.

Propo-

Propoſition II.

To draw a Tangent to a Parabola.

VA $= x$.
The Parameter $= p$, and the Property of a *Parabola*. } Given.
AT $= t$ Required. *Note*, AD ⊥ VF.



Preparation.

Produce the Tangent TD to H, and imagine the Line HGF to be drawn ‖, and indefinitely near DA, and LKI ‖, and at the ſame Diſtance from DA; and ſuppoſe the indefinitely-ſhort Line IA $= a =$ AF.

Solution.

By the Property of a *Parabola* $\left\{\begin{array}{l} px = ADq: \\ px + pa = GFq. \end{array}\right.$

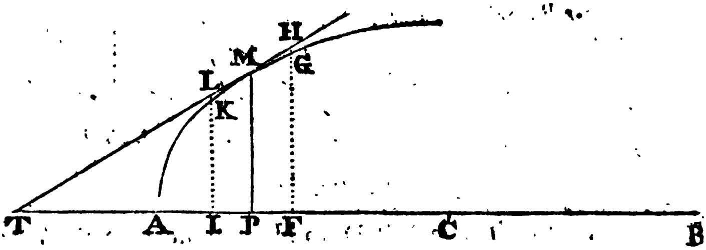
The $\triangle s$ ADT and FHT are ſimilar; therefore $t \cdot\cdot t + a :: \sqrt{px} \cdot\cdot HF$; conſequently $tt\ (ATq:) \cdot\cdot tt + 2ta + aa\ (TFq) :: px\ (ADq = p \times VA) \cdot\cdot HFq = px + \frac{2tpxa + pxaa}{tt}$: But $HFq: \sqsubset GFq$, that is $px + \frac{2tpxa + pxaa}{tt} \sqsubset px + pa$; therefore $2tpxa + pxaa \sqsubset + ptta$, and $2x + \frac{ax}{t} \sqsubset t$.

Again, $px - pa = KIq$ (by the Property of a *Parabola*:) And ſince the $\triangle s$ TAD and TIL are ſimilar; $t \cdot\cdot t - a :: \sqrt{px} \cdot\cdot IL$; wherefore $tt \cdot\cdot tt - 2ta + aa :: px \cdot\cdot ILq: = px - \frac{2tapx + aapx}{tt}$; But $ILq: \sqsubset IKq$; that is, $px - \frac{2tpxa + pxaa}{tt} \sqsubset px - pa$; wherefore $-2tpxa + pxaa \sqsubset -ttpa$; wherefore $t + \frac{xa}{t} \sqsubset 2x$. That is, by the laſt Step

$t + \frac{xa}{t} \sqsubset 2x$, and by one of the foregoing Steps, $2x + \frac{xa}{t} \sqsubset t$; conſequently (by our *Lemma* 1.) $t = 2x$.

Propoſition III.

To draw a Tangent to an Ellipſis.



The Parameter $= p$.
AB $= b$.
AP $= x$; then PB $= b - x$, and the Property of an *Ellipſis* } Given.
Required to find PT $= t = ?$ *Note,* PM $\perp$ AB.

Preparation.

Produce the Tangent TM to H, and imagine the Lines HGF and LKI to be drawn at an indefinitely little equal Diſtance from, and parallel to MP; and ſuppoſe the indefinitely ſhort Line IP $=$ PF $= a$.

Solution.

By the Property of an *Ellipſe* $x \times \overline{b - x} \cdot\cdot \text{PM}q :: b \cdot\cdot p$; conſequently PM$q$: $= \frac{pbx - pxx}{b}$.

In like manner will be found $\frac{\overline{x + a} \times \overline{b - x - a} \times p}{b} = \text{FG}q = \frac{pbx - pxx - 2xap + abp - aap}{b}$; as alſo $\frac{\overline{x - a} \times \overline{b - x + a} \times p}{b} = \text{IK}q: = \frac{pbx - pxx + 2pxa - bpa - aap}{b}$.

The $\triangle$s TFH, TPM, TIL are ſimilar; wherefore tt (TPq:) .. $\frac{pbx - pxx}{b}$ (PMq:) :: $tt + 2ta + aa$ (FTq) .. FHq
$= \frac{ttpbx + 2tapbx + aapbx - pxxtt - 2tapx^2 - aapxx}{ttb}$.

And tt (TPq:) .. $\frac{pbx - pxx}{b}$ (PMq:) :: $tt - 2ta + aa$
(TIq:) .. ILq:
$= \frac{pbxtt - pxxtt - 2tapbx + 2tapxx + a^2pbx - a^2px^2}{ttb}$

But FHq: $\sqsubset$ FGq;
that is, $\frac{ttbpx + 2tapbx + aapbx - pxxtt - 2tapxx - aapxx}{ttb}$
$\sqsubset \frac{xbp - xxp - 2axp + abp - aap}{b}$, which Step, after due Reduction, gives $2bx - 2xx + \frac{abx + tta - axx}{t} \sqsubset tb - 2tx$; that is (becauſe b is $\sqsubset x$, and of Conſequence * $abx + tta \sqsubset axx$) $2bx - 2xx \sqsubset tb - 2tx$.

Again, ILq: $\sqsubset$ IKq:,
that is, $\frac{pbxtt - px^2t^2 - 2tapbx + 2tapxx + a^2pbx - a^2pxx}{ttb}$
$\sqsubset \frac{pbx - pxx + 2pxa - abp - aap}{b}$; which Step, when duly reduc'd, gives $tb - 2tx + \overline{\frac{bx + tt - xx}{t}} \times a \sqsubset 2bx - 2x^2$; that is, $tb - 2tx \sqsubset 2bx - 2xx$.

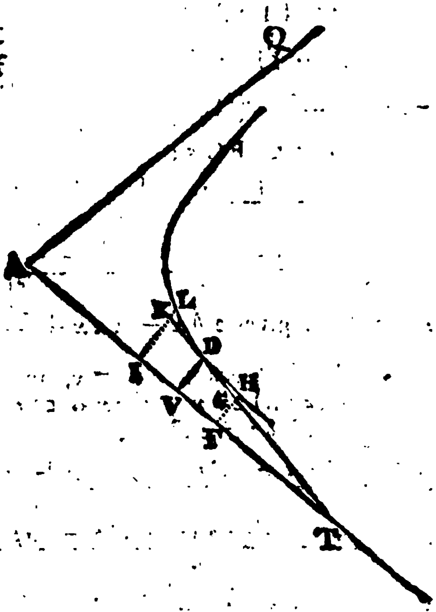
Now from this laſt Step, and from that mark'd with * an Aſterick, it muſt follow (by our *Lemma* 1.) that $tb - 2tx = 2bx - 2x^2$: Whence $t = \frac{2bx - 2xx}{b - 2x}$.

*From the * Coherence that is in, or between the* Ellipſis *and* Hyperbola, *I preſume it very eaſy to draw a Tangent to an* Hyperbola, *by the like Method with that us'd in* Prop. 3. *I will therefore leave it to the* Reader's *Practice, and here only ſhew, how, from the Aſymptotes of an* Hyperbola, *to draw a Tangent to its Curve.*

* *See* Apollonius's *Conicks.*

Propoſition IV.

To draw ſuch a Tangent as is abovemention'd to an Hyperbola.



Let AQ and AT repreſent the *Aſymptotes* of the *Hyperbola* LDH.

VA = x,
And the Proportion of any Line as VD being ⊥ AV, to AV, being given.

Required to find VT = t.

Note, p being = Parameter that, by the Property of an *Hyperbola*, pp = AV × VD; But here, for ſome Reaſons, I will (as I may) ſuppoſe AV = x = $\overline{VD}|^{-m}$ Then

Preparation.

Produce the Tangent TD to L, and imagine the Lines LKI and HGF to be drawn at an a, or indefinitely-little equal Diſtance from, and ∥ VD; then

Solu-

Solution.

x being (by Suppoſition) $= \overline{\mathrm{VD}}\big|^{-m} = \frac{1}{\overline{\mathrm{VD}}|^{m}}$; 1 is therefore $= x \times \overline{\mathrm{VD}}\big|^{m}$ and conſequently $\overline{\frac{1}{x}}\Big|^{\frac{1}{m}} = \mathrm{VD}$.

Alſo $\mathrm{AI} = \overline{x - a} =$ (by our ſuppos'd Property of the *Hyperbola*) $\overline{\mathrm{IL}}\big|^{-m}$; therefore $\overline{\frac{1}{x-a}}\Big|^{\frac{1}{m}} = \mathrm{LI}$: And for the like Reaſons HF is $= \overline{\frac{1}{x+a}}\Big|^{\frac{1}{m}}$.

The Δs TDV, TGF, TKI are ſimilar; wherefore $t : \overline{\frac{1}{x}}\Big|^{\frac{1}{m}} :: t - a : \overline{1 - \frac{a}{t}} \times \overline{\frac{1}{x}}\Big|^{\frac{1}{m}} = \mathrm{FG} :: t + a : \overline{1 + \frac{a}{t}} \times \overline{\frac{1}{x}}\Big|^{\frac{1}{m}} = \mathrm{IK}$; but $\mathrm{FH} \sqsubset \mathrm{FG}$; that is, $\overline{\frac{1}{x+a}}\Big|^{\frac{1}{m}} \sqsubset \overline{1 - \frac{a}{t}} \times \overline{\frac{1}{x}}\Big|^{\frac{1}{m}}$; and by dividing 1 by $x + a$, you have $\overline{\frac{1}{x} - \frac{a}{xx} + \frac{a^2}{x^3} \bullet}\Big|^{\frac{1}{m}} \sqsubset \overline{1 - \frac{a}{t}} \times \overline{\frac{1}{x}}\Big|^{\frac{1}{m}}$. And by raiſing each Part to the mth-Power, you have $\frac{1}{x} - \frac{a}{xx} + \frac{a^2}{xxx} \bullet \sqsubset \overline{1 - m \times \frac{a}{t} + \frac{m^2 - m}{2} \times \frac{aa}{tt} \bullet} \times \frac{1}{x}$, which Step, when reduc'd, gives $\frac{m}{t} + \frac{a}{x^2} - \overline{\frac{m^2 - m}{2}} \times \frac{a}{tt} \bullet \sqsubset \frac{1}{x}$.

Again, $\mathrm{IL} \sqsubset \mathrm{IK}$; that is $\overline{\frac{1}{x-a}}\Big|^{\frac{1}{m}} \sqsubset \overline{1 + \frac{a}{t}} \times \overline{\frac{1}{x}}\Big|^{\frac{1}{m}}$; that is, by Diviſion, $\overline{\frac{1}{x} + \frac{a}{xx} + \frac{a^2}{x^3} \bullet}\Big|^{\frac{1}{m}} \sqsubset \overline{1 + \frac{a}{t}} \times \overline{\frac{1}{x}}\Big|^{\frac{1}{m}}$: And by equal Involution you have $\frac{1}{x} + \frac{a}{x^2} + \frac{a^2}{x^3} \bullet \sqsubset \overline{1 + m \times \frac{a}{t} + \frac{m^2 - m}{2} \times \frac{aa}{tt} \bullet} \times \frac{1}{x}$, which Step, after due Reduction, gives $\frac{1}{x} + \frac{a}{x^2} - \overline{\frac{m^2 - m}{2}} \times \frac{a}{tt} \bullet \sqsubset \frac{m}{t}$. From which, and from the Step at * it is manifeſt:

Firſt, That $\frac{a}{xx} \sqsubset \frac{m^2 - m}{2} \times \frac{a}{tt} \bullet$, And

Secondly, That $\frac{1}{x}$ is (by our *Lemma* 1.) $= \frac{m}{t}$; conſequently $t = mx$.

Note, *In theſe Operations be ſure to take care that your Multiplicator be always* $\sqsubset 0$, *otherwiſe the Majority will not hold, as in the foregoing Step: Thus, If* $a \sqsubset b$, *and you multiply each by* c; *if* $c \sqsubset 0$, *then* $ca \sqsubset cb$; *but if* $c \sqsupset 0$, *then* $ca \sqsupset cb$.

CHAP. II.

De Maximis & Minimis.

Propoſition I.

DIvide b into two ſuch Parts, that their Product will be a *Maximum.*

Preparation.

Suppoſe $x =$ one Part, then $b - x =$ the other Part of b, and then $\overline{b-x} \times x = bx - xx =$ *Maximum.*

Solution.

1*ſt*, Suppoſe the Part x to be increaſed by the indefinitely little, or leſs than any aſſignable Quantity a, then the other Part of b will be $b - x - a$, and $\overline{x+a} \times \overline{b-x-a} = bx - xx - 2xa + ab - aa$.

But becauſe $bx - xx$, is (by Suppoſition) a *Maximum*, therefore $bx - xx \sqsubset bx - xx - 2xa + ab - aa$:

And $o \sqsubset - 2xa + ab - aa$:
And $2x + a \sqsubset b$.

2*dly*, Suppoſe the part x to be diminiſh'd by the indefinitely ſmall Quantity a, then the other part of b will be $b - x + a$;
And $\overline{x-a} \times \overline{b-x+a} = xb - xx + 2xa - ba - aa$.

But ſince $bx - xx$ is (by Suppoſition) a *Maximum*; therefore $bx - xx \sqsubset xb - xx + 2xa - ba - aa$. Whence $o \sqsubset 2xa - ba - aa$, and $b + a \sqsubset 2x$.

Now it appears, by the latter parts of the firſt and ſecond Suppoſitions, that, let a be ever ſo ſmall a Quantity (provided it be more than nothing) if it be added to $2x$, the Sum will be greater than b; but if it be added to b, then the Sum will be greater than $2x$; conſequently (by our *Lemma* 1.)

$$2x = b,$$

$$\text{And } x = \frac{b}{2}$$

Propoſition II.

Divide b into three ſuch Parts, as being multiply'd together ſhall produce a *Maximum.*

Suppoſe $b - x$ to be one of the Parts requir'd, then x will be $=$ the Sum of the other two Parts; but the greateſt Product that can be made by any two parts of x, is (by the foregoing *Propoſition*) $= \frac{x}{2} \times \frac{x}{2}$; the Queſtion propos'd is therefore reduc'd to this, *viz.*

$\frac{x}{2} \times \frac{x}{2} \times \overline{b - x} = \frac{bx^2 - x^3}{4} =$ *Maximum.* Quere x; it being $=$ an affirmative Quantity.

1*ſt*, Suppoſe x to be encreas'd by the indefinitely-little Quantity a, then $\frac{bx^2 - x^3}{4}$ will become $\frac{x + a}{2} \times \frac{x + a}{2} \times \overline{b - x - a}$: But the former being a *Maximum*, is therefore greater than the latter; that is $\frac{bx^2 - x^3}{4} \sqsubset \frac{bx^2 - x^3 + 2bxa + ba^2 - 3x^2a - 3xa^2 - a^3}{4}$ Wherefore $0 \sqsubset \frac{2bxa + ba^2 - 3x^2a - 3xa^2 - a^3}{4}$: And by multiplying each part by $\frac{4}{a}$, and tranſpoſing) $3x^2 + 3xa - ba \sqsubset 2bx$.

2*dly*, Suppoſe x to be diminiſh'd by the indefinitely ſmall Quantity a; then $\frac{bx^2 - x^3}{4}$ will become $\frac{\overline{x - a}}{2} \times \frac{\overline{x - a}}{2} \times \overline{b - x + a}$: But the former being a *Maximum*, is therefore greater than the latter; that is, $\frac{bx^2 - x^3}{4}$ $\sqsubset \frac{bx^2 - x^3 - 2bxa + ba^2 + 3x^2a - 3xa^2 + a^3}{4}$: Conſequently $2bx + 3xa - ba \sqsubset 3x^2$; whence, and from the latter Part of the firſt Suppoſition, $3xa \sqsubset ba$; and $2bx$ is (by our *Lemma* 1.) $= 3x^2$; wherefore $2b = 3x$; and $\frac{2}{3}b = x$.

Prop.

Propoſition III.

If n be $\sqsubset m$, and $x^m - x^n = $ *Maximum*; It is requir'd to find the Value of x, it being affirmative.

Suppoſe x to be augmented by the indefinitely - little Quantity a: Then $x^m - x^n$ will become $\overline{x+a}|^m - \overline{x+a}|^n$; but the former being a *Maximum*, is therefore greater than the latter; that is (by Sir *Iſaac Newton*'s Theorem) $x^m - x^n \sqsubset x^m + mx^{m-1}a + m \times \frac{m-1}{2} x^{m-2}a^2 \bullet - x^n - nx^{n-1}a - n \times \frac{n-1}{2} x^{n-2}a^2 \bullet$: Wherefore $o \sqsubset mx^{m-1}a + \frac{m^2-m}{2} x^{m-2}a^2 \bullet - nx^{n-1}a - \frac{n^2-n}{2} x^{n-2}aa \bullet$: And (by dividing each part by a, and tranſpoſing) $nx^{n-1} + \frac{n^2-n}{2} x^{n-2}a - \frac{m^2-m}{2} x^{m-2}a \bullet \sqsubset mx^{m-1}$.

Suppoſe x to be decreas'd by the indefinitely - little Quantity a; then $x^m - x^n$ will become $\overline{x-a}|^m - \overline{x-a}|^n$: But the former being a *Maximum*, is greater than the latter; that is $x^m - x^n \sqsubset x^m - mx^{m-1}a + \frac{m^2-m}{2} \times x^{m-2}aa \bullet - x^n + nx^{n-1}a - \frac{n^2-n}{2} x^{n-2}aa \bullet$; and (by tranſpoſing and dividing each part by a) $mx^{m-1} + \frac{n^2-n}{2} x^{n-2}a - \frac{m^2-m}{2} x^{m-2}a\bullet \sqsubset nx^{n-1}$.

From the latter parts of the firſt and ſecond Suppoſitions it is plain, that the Difference of nx^{n-1} and mx^{m-1}; and therefore of $\frac{n}{2} x^{n-2}a$; and $\frac{m}{2} x^{m-2}a$, is at moſt but $\bullet$; wherefore (ſince n is $\sqsupset m$) $\frac{n^2}{2} x^{n-2}a \sqsubset \frac{m^2}{2} x^{m-2}a \bullet$: And (from the two laſt Steps) $\frac{n^2-n}{2}x^{n-2}a \sqsubset \frac{m^2-m}{2}x^{m-2}a$.

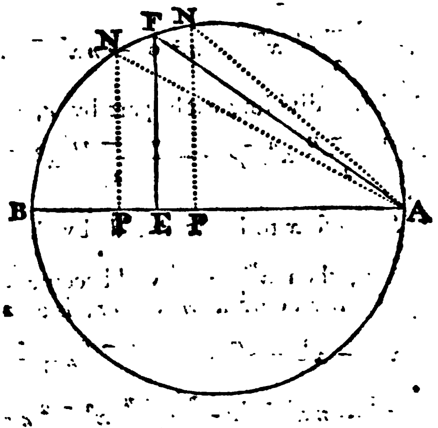
Hence, and from the ſaid latter Parts of the firſt and ſecond Suppoſitions nx^{n-1} is (by our *Lem.* 1.) $= mx^{m-1}$:

Conſequently $x = \overline{\frac{m}{n}}\Big|^{\frac{1}{n-m}}$

Propoſition IV.

Of all the *Cones* that can be inſcrib'd in a *Given Sphere*; 'tis requir'd to find that which has the greateſt Convex Surface.

This *Propoſition* amounts to no more than this; to determine the Point E in the Diameter A B of the Circle A B E, ſo that the *Rectangle* comprehended under A F and F E, be the greateſt of all the like *Rectangles* comprehended under A N and N P.



For, if we imagine the Semicircle A F B to revolve about its Axis A B, it is evident that the Semicircle deſcribes a Sphere, and the ⌞d △s A F E and A N P deſcribe Cones inſcrib'd in the ſame Sphere, whoſe Surfaces are proportional to the reſpective Rectangles A F × F E, and A N × N P.

Suppoſe the unknown Quantity $AE = x$, AB (the Diameter of the Sphere) $= 2r$: And the indefinitely-ſhort-Line $PE = EP = a$: Then (by the Property of a Circle) $\overline{2rx - xx}|^{\frac{1}{2}} = EF$, and $\overline{2rx - xx + xx}|^{\frac{1}{2}} = \overline{2rx}|^{\frac{1}{2}} = FA$; and $\overline{2rx}|^{\frac{1}{2}} \times \overline{2rx - xx}|^{\frac{1}{2}} = \overline{4r^2x^2 - 2rx^3}|^{\frac{1}{2}} =$ *Maximum*. Quære x, it being $=$ an affirmative Quantity.

Suppoſe x (AE) to be increas'd by the indefinitely-little Quantity a (PE) then $\overline{4r^2x^2 - 2rx^3}|^{\frac{1}{2}}$ (= AF × FE) will become $\overline{4r^2x^2 - 2rx^3 - 6rx^2a + 8r^2xa - 6rxa^2 + 4r^2a^2 - 2ra^3}|^{\frac{1}{2}}$ (= AN × PN;) but the former being a *Maximum*, is therefore greater than the latter; and conſequently the Square of the former, *viz.* $4r^2x^2 - 2rx^3$ is $\sqsubset$, the Square of the latter, *viz.* $4r^2x^2 - 2rx^3 - 6rx^2a + 8r^2xa - 6rxa^2 + 4r^2a^2 - 2ra^3$. Wherefore $6rxx + 6rxa - 4r^2a \circ \sqsubset 8r^2x$.

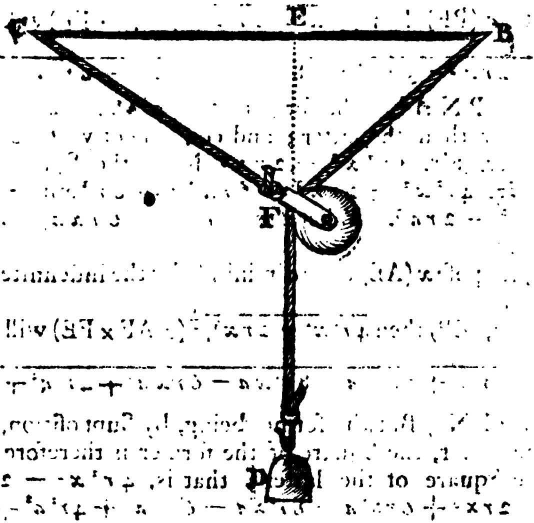
Again, ſuppoſe x (AE) to be diminiſh'd by the indefinitely-little Quantity a, (EP) then $\overline{4r^2x^2 - 2rx^3}|^{\frac{1}{2}}$ (= AF × FE) will become $\overline{4r^2x^2 - 2rx^3 + 6rx^2a - 8r^2xa - 6rxa^2 + 4r^2a^2 + 2ra^3}|^{\frac{1}{2}}$ (= AN × PN:) But the former being, by Suppoſition, greater than the latter, the Square of the former is therefore greater than the Square of the latter; that is, $4r^2x^2 - 2rx^3 \sqsubset 4r^2x^2 - 2rx^3 + 6rx^2a - 8r^2xa - 6rxa^2 + 4r^2a^2 + 2ra^3$; wherefore $8r^2x + 6rxa - 4r^2a \circ \sqsubset 6rx^2$.

From what has been ſaid it is evident, that 1*ſt*, $6rxa \sqsubset 4r^2a \circ$; 2*dly*, (by our *Lem.* 1.) $8r^2x = 6rx^2$; and $8r = 6x$; and *laſtly*, $\frac{8}{6}r = \frac{4}{3}r = x = r + \frac{1}{3}r$.

Note, *If this* Queſtion *had been propos'd*, viz. $x^2 - x$ *being* = *Maximum*; *what is* x = *to?*

Here 'tis evident, that by how much the greater you ſuppoſe the Value of x to be, by ſo much the nearer you come to it; therefore it is not to be determin'd, unleſs you ſuppoſe, that Infinity is ſufficiently great; then x may be $= \frac{1}{0}$: But if any Number can be aſſign'd that is greater than $\frac{1}{0}$, as $\frac{2}{0}$, $\frac{3}{0}$ or &c. (for really I do not know which are they Equal, or retain the ſame Proportion as their Numerators) then the Value of x is not to be aſſign'd: And after all, a Negative ſeems to go nearer to give the Value of x than the ſame Number if Affirmative: And therefore I have all along exempted Negatives, as the very Operations do. But if this Queſtion had been propos'd, *viz.* $x^2 - x =$ *Minimum*; then the Value of x will be found (by the foregoing Method) $= \frac{1}{2}$.

Propoſition V.



Let C F be a Chord, and let one End thereof be made faſt at C, and to the other end faſten the Pulley F, about which ſuſpend the Weight D by the Chord D F B, faſtening one end thereof at B, and let the Points B and C be in the ſame Horizontal Line C B: And ſuppoſe both the Chords and the Pulley to be without * Weight; 'tis requir'd to find the loweſt Deſcent of the Pulley, and of the Weight.

It is evident, that the Weight D will deſcend below the Horizontal Line C B, as low as the Chords C F and B F D will permit, and therefore the Line D F E will repreſent the greateſt Deſcent of the Weight D.

Suppoſe the known Quantities $DFB = b$, $CB = c$, and $CF = d$: And ſuppoſe the unknown and variable Quantity $CE = x$; then is $EF = \sqrt{d^2 - x^2}$, and $FB = \overline{dd + cc - 2cx}|^{\frac{1}{2}}$; and $DFE = b - \overline{d^2 + c^2 - 2cx}|^{\frac{1}{2}} + \overline{d^2 - xx}|^{\frac{1}{2}} =$ *Maximum*. Quere, x.

---

* Note, *What is meant by the* Chords *and* Pulleys *being without Weight, is, that they are of no Weight in reſpect of the Weight* D: *For if that be not ſuppos'd, the Weight of the ſaid* Chords *and* Pulley *may be a means to hinder the Weight* D*'s deſcending as low as the* Chords *will permit.*

Solution.

1ſt, Suppoſe x to be augmented by the indefinitely-little Quantity a; then $b - \overline{d^2 + c^2 - 2cx}|^{\frac{1}{2}} + \overline{d^2 - x^2}|^{\frac{1}{2}}$ will become $b - \overline{d^2 + c^2 - 2cx - 2ca}|^{\frac{1}{2}} + \overline{d^2 - x^2 - 2xa - aa}|^{\frac{1}{2}}$: But ſince the former is a *Maximum*, it is, of Conſequence, greater than the latter; that is $b - \overline{d^2 + c^2 - 2cx}|^{\frac{1}{2}} + \overline{d^2 - x^2}|^{\frac{1}{2}} \sqsubset b - \overline{d^2 + c^2 - 2cx - 2ca}|^{\frac{1}{2}} + \overline{d^2 - x^2 - 2xa - aa}|^{\frac{1}{2}}$ that is (by Evolution) $b - \overline{d^2 + c^2 - 2cx}|^{\frac{1}{2}} + \overline{d^2 - x^2}|^{\frac{1}{2}} \sqsubset b - \overline{d^2 + c^2 - 2cx}|^{\frac{1}{2}} - \frac{1}{2} \times \overline{d^2 + c^2 - 2cx}|^{\frac{1}{2} - 1} \times -2ca - \frac{1}{2} \times \frac{\frac{1}{2} - 1}{2} \times \overline{d^2 + c^2 - 2cx}|^{\frac{1}{2} - 2} \times \overline{-2ca}|^{2}\,\phi + \overline{d^2 - x^2}|^{\frac{1}{2}} + \frac{1}{2} \times \overline{d^2 - x^2}|^{\frac{1}{2} - 1} \times \overline{-2xa - aa} + \frac{1}{2} \times \frac{\frac{1}{2} - 1}{2} \times \overline{d^2 - x^2}|^{\frac{1}{2} - 2} \times \overline{-2xa - aa}|^{2}\,\phi$. Which laſt Step reduc'd gives $\frac{x + \frac{1}{2}a}{\sqrt{dd - xx}} + \frac{xxa}{2 \times \overline{d^2 - x^2}|^{\frac{3}{2}}} - \frac{cca}{2 \times \overline{d^2 + c^2 - 2cx}|^{\frac{3}{2}}}\,\phi \sqsubset \frac{c}{\sqrt{d^2 + c^2 - 2cx}}$.

2dly, Suppoſe x to be diminiſh'd by the indefinitely-little Quantity a; then $b - \overline{d^2 + c^2 - 2cx}|^{\frac{1}{2}} + \overline{d^2 - x^2}|^{\frac{1}{2}}$ will become $b - \overline{d^2 + c^2 - 2cx + 2ca}|^{\frac{1}{2}} + \overline{d^2 - x^2 + 2xa - aa}|^{\frac{1}{2}}$: But the former is a *Maximum*, and muſt therefore be greater than the latter; whence it will follow, that $\frac{c}{\sqrt{dd + cc - 2cx}} + \frac{x^2a}{2 \times \overline{d^2 - x^2}|^{\frac{3}{2}}} - \frac{c^2a}{2 \times \overline{d^2 + c^2 - 2cx}|^{\frac{3}{2}}} \sqsubset \frac{x - \frac{1}{2}a}{\sqrt{dd - xx}}$.

From the latter Parts of the firſt and ſecond Suppoſitions it is manifeſt, that the Difference of $\frac{c}{\sqrt{dd + cc - 2cx}}$, and $\frac{x}{\sqrt{dd-xx}}$; and conſequently of half the Cube of each $\times\, a$, *viz.* of $\frac{c^3 a}{2\times\overline{d^2+c^2-2cx}|^{\frac{3}{2}}}$ and $\frac{x^3 a}{2\times\overline{d^2-x^2}|^{\frac{3}{2}}}$ is at moſt but o; wherefore (ſince c is $\sqsubset x$)

$$\frac{x^2 a}{2\times\overline{d^2-x^2}|^{\frac{3}{2}}} \sqsubset \frac{c^2 a}{2\times\overline{d^2+c^2-2cx}|^{\frac{3}{2}}}\ o$$; and conſequently

$$\frac{xxa}{2\times\overline{d^2-x^2}|^{\frac{3}{2}}} + \frac{\frac{1}{2}a}{\sqrt{dd-xx}} \sqsubset \frac{cca}{2\times\overline{dd+cc-2cx}|^{\frac{3}{2}}}\ o;$$

Hence, and from the ſaid latter Parts of the firſt and ſecond Suppoſitions (by our *Lem.* 1.) $\frac{c}{\sqrt{dd+cc-2cx}} = \frac{x}{\sqrt{dd-xx}}$; wherefore $ccdd - ccxx = ddxx + ccxx - 2cx^3$; that is, $2cx^3 - 2c^2x^2 - d^2x^2 + c^2d^2 = o$. Finally, by dividing each part by $\overline{x-c}$ you'll have $2cx^2 - d^2x - cd^2 = o$.

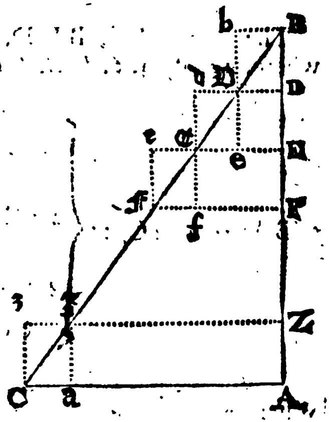
CHAP. III.

Of *Quadratures*.

Propoſition and Problem I.

GIven the Baſe A C, and Perpendicular A B of the ∟d △ A B C; required to find its *Area*.

A C = *b*
B A = *p* } Given.
Area of ∟*d* △ A B C = [illegible]
Required.



Preparation.

Supppoſe the ⊥ B A to be divided into an indefinite Number of equal Parts B D, D E, E F, &c. and Z A; and ſuppoſe each of thoſe Parts = *a*; then, through the Points B, D, E, F, &c. and Z, draw the Lines B b, D 𝔇 𝔡, E e 𝔈 𝔢, F f 𝔉, &c. and Z ℨ 𝔷 ∥*s* A C, and thro' the Points 𝔇, 𝔈, 𝔉, &c. ℨ and 𝔷. Draw the Lines b 𝔇 e, 𝔡 𝔈 f, 𝔢 𝔉, &c. ℨ a and 𝔷 C ∥*s* B A; then the Sum of the ▭*s* B b D 𝔇, D 𝔡 𝔈 E, E 𝔢 𝔉 F, &c. and Z 𝔷 C A is called the Sum of the † Circumſcribing ▭*s*; and the Sum of the ▭*s* D 𝔇 e E, E 𝔈 f F, &c. and Z ℨ a A is call'd the Sum of the inſcrib'd ▭*s*. Now the firſt, (or leaſt) ſecond, third, &c. of the Circumſcribing ▭*s* being = firſt, ſecond, third, &c. of the inſcrib'd

Note, *I ſuppoſe it known, that the* Area *of any* ▭ *is had by multiplying the length by its Perpendicular Height.*

† *Becauſe their Sum circumſcribes the* △.

ſcrib'd ▭ *s* reſpectively, and the Number of the Circumſcribing ▭ *s* being (as manifeſtly it is) = Number of equal Parts into which A B is, by Suppoſition, divided $= \frac{BA}{a}$ $(= \frac{p}{a})$ and by one more than the inſcrib'd ▭'*s* Number; the Sum therefore of the *Area's* of the Circumſcribing ▭ *s* is, by the *Area* of the greateſt of them, more than the Sum of the *Area's* of the inſcrib'd ▭ *s*.

Solution.

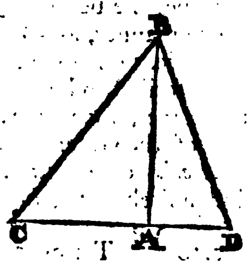
The △ *s* B A C, B D $\mathfrak{D}$, B E $\mathfrak{E}$, B F $\mathfrak{F}$, *&c.* are ſimilar; wherefore (by 4. 6. *Eucl. El.*)

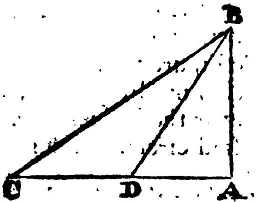
$$p\,(BA) \,..\, b\,(AC) :: \begin{cases} a\,(BD) \,..\, \frac{ba}{p} = D\mathfrak{D} = Bb \\ 2a\,(BE) \,..\, \frac{2ba}{p} = E\mathfrak{E} = D\mathfrak{d}. \\ 3a\,(BF) \,..\, \frac{3ba}{p} = F\mathfrak{F} = E\mathfrak{e}. \\ \text{\&c.} \end{cases}$$

Wherefore $\frac{ba}{p} \times a \times \overline{1 + 2 + 3 + 4 + \text{\&c. continu'd to } \frac{p}{a}}$ $\overline{\text{Terms}} = \frac{ba}{p} \times a \times \overline{\frac{p}{a} + 1} \times \frac{p}{2a}$ (by our *Lemma* 2. or by Part 13 or 14) $= \frac{bp + ba}{2}$ is = to the Sum of the *Area's* of the Circumſcribing ▭ *s*; and therefore ⊏ $\mathfrak{A}$; but $\frac{bp + ba}{2} - ba = \frac{bp - ba}{2}$ is = the Sum of the *Area's* of the inſcrib'd ▭ *s*, and therefore ⊐ $\mathfrak{A}$; wherefore (by our *Lemma* 1.) $\frac{bp}{2}$ is = $\mathfrak{A}$.

Scholia.

1. Hence the *Area* of any △ BCD may be had thus, *viz.*





The *Area's* of the ∟*d* △ BCA is $= \frac{CA \times BA}{2}$, *See Fig.* 1 *and* 2. and of the ∟*d* △ BAD is $= \frac{DA \times BA}{2}$ (by the foregoing Propoſition ;) wherefore the *Area* of the △ BCD is $= \frac{CA \times BA}{2} \pm \left\{ \textit{Fig.} \left\{ \begin{matrix} 1 \\ 2 \end{matrix} \right\} \right. \frac{DA \times BA}{2} = \frac{CA \pm \textit{Fig.} \left\{ \begin{matrix} 1 \\ 2 \end{matrix} \right\} DA \times BA}{2} = \frac{CD \times BA}{2}$ &c.

2. 'Tis plain that if ▭*s* of equal Baſes be inſcrib'd in, or circumſcrib'd about any Plane Figure terminating at one end in a Point, and thence towards the other end continually increaſing, that the Number of the circumſcribing ▭*s* is = to the Number of equal Parts into which the proper Line is ſuppos'd to be divided, and by one more than the Number of the inſcrib'd ▭*s*.

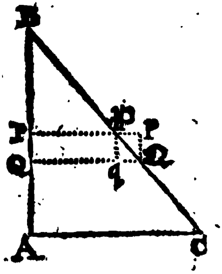
3. But if ▭*s* of equal Baſes be ſuppos'd to be inſcrib'd in, and circumſcrib'd about any Figure, neither of whoſe Ends terminates in a Point (as CAF &c.) [*See Fig. to Prop.* 1.] Or if △*s* or Right Lines be inſcrib'd in, and circumſcrib'd about any Figure whatſoever, the Number of the circumſcribing ▭*s*, △*s*, or Right Lines is = to the Number of equal parts into which the proper Line is ſuppos'd to be divided, as alſo equal the Number of the inſcrib'd ▭*s*, △*s*, or |*s*.

4. [*Obſerve here.*] But, if inſtead of circumſcribing and inſcribing the given Figure with indefinitely-little Figures,

In the foregoing manner you only ſuppoſe the proper Line; ſuppoſe AB to be divided into an indefinite Number of equal Parts, each Part being $= a$, and through any two of thoſe Diviſions next to one another at the Diſtances za; and $\overline{z-1} \times a$ from (what you make to be) the beginning of the Line AB you draw two Lines, in ſuch manner, as the Nature of the *Problem* requires, and then compleat one of the circumſcribing along with its reſpective inſcrib'd indefinitely-little Figure: Then having drawn other Lines, ſuch as are (if any be) needful, you'll, by the Property of the Given Figure, diſcover the Values (*i. e.* the *Lengths, Area's,* or *Solidities*) of the ſaid circumſcribing and inſcrib'd indefinitely-little Figures ſeverally; then, by placing 1, 2, 3, 4, &c. to $\frac{AB}{a}$ Terms continu'd, inſtead of z, and Univerſally $1^n, 2^n, 3^n, 4^n$ &c. inſtead of z^n, in each of the ſaid Values, you will have the Values of the circumſcribing and inſcrib'd indefinitely-little Figures ſeverally, in two or more Series, whoſe Sums will be had exactly, or *proxime*, by our *Lemma* 2. (or by Part 13 or 14.) whence (by our *Lemma* 1.) the Value of the propos'd Figure will be found.

By the help of this Schol. 4. Prop. 1. *may be ſolv'd thus,*

$AC = b$ } Given.
$BA = p$ }
Area of $\llcorner d \triangle ABC = \mathfrak{Z}$ required.



Suppoſe the $\perp$ BA to be divided into an indefinite Number of equal Parts, each part being $= a$ ($\doteq$ PQ;) then, at the indetermin'd Diſtances $BQ = za$, and $BP = \overline{z-1} \times a$ from B draw the Lines Qq$\mathfrak{Q}$, and P$\mathfrak{P}$p $\perp s$ BA, and draw the Lines $\mathfrak{P}$q, p$\mathfrak{Q}$ $\parallel s$ BA: Then the $\triangle s$ BAC, BQ$\mathfrak{Q}$, and BP$\mathfrak{P}$ are ſimilar; wherefore

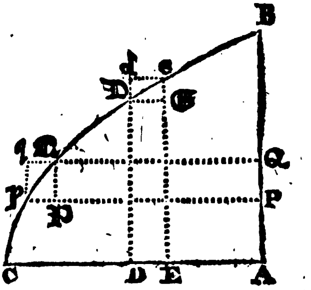
$$p\,(AB) \cdot\cdot\, b\,(AC) :: \begin{cases} za\,(BQ) \cdot\cdot\, \frac{zba}{p} = Q\mathfrak{Q} = Pp. \\ \overline{z-1} \times a\,(BP) \cdot\cdot\, \frac{zba-ba}{p} = P\mathfrak{P} \\ = Qq. \end{cases}$$

Wherefore $\frac{zba}{p} \times a$ is $=$ *Area* of $\square\, Pp\mathfrak{Q}Q$, and $\frac{\overline{zba-ba}}{p}$ $\times a$ is $=$ *Area* of $\square\, P\mathfrak{P}qQ$; wherefore (by *Scholium* 4) $\frac{ba^2}{p}$ $\times \overline{1+2+3+4+\&c.}$ continued to $\frac{p}{a}$ Terms $= \frac{bp+ba}{2}$ (by our *Lem.* 2, or * by Part 13 or 14.) is $=$ Sum of the *Area's* of the circumſcribing $\square$*s*, and therefore $\sqsubset \mathfrak{A}$: And $\frac{a^2b}{p} \times \overline{0+1+2+3+\&c.}$ continu'd to $\frac{p}{a}$ Terms $= \frac{bp-ba}{2}$ (by our *Lemma* 2. or by Part 13 or 14.) $=$ the Sum of the *Area's* of the inſcrib'd $\square$*s*, and therefore $\sqsupset \mathfrak{A}$; conſequently (by our *Lemma* 1.) $\frac{bp}{2} = \mathfrak{A}$.

* Book I.

Problem II.

Given the *Abſciſſe* CA, Ordinate AB, and property of the *Semi-Parabola* ABC: Required to find the *Area* of the ſaid *Semi-parabola* ABC.



Suppoſe the Ordinate AB to be divided into an indefinite Number of equal Parts, each Part being $= a$ $(= PQ)$ and through any two next to one another of thoſe Diviſions, as Q and P at the Diſtances $za = AQ$, and $\overline{z-1} \times a = AP$ from A, draw the Lines $Q\mathfrak{Q}q$ and $P\mathfrak{P}p$ $\perp$*s* AB (or $\parallel$*s* AC;) and draw the Lines qp and $\mathfrak{Q}\mathfrak{P}$ $\parallel$*s* AB.

Now ſuppoſe $AB = z$, $AC = x$, and $\overline{AQ}|^m = AC - \mathfrak{Q}Q$. Then

Solution.

By the Property of the *Parabola* $\overline{x - za}|^m$ (= $\overline{AC - AQ}|^m$) = qQ; and $\overline{x - \overline{z - 1} \times a}|^m$ (= $\overline{AC - AP}|^m$) = pP. Whence $a \times \overline{x - za}|^m$ is = *Area* of the inſcrib'd □ PpqQ: And $a \times \overline{x - \overline{z - 1} \times a}|^m$ is = *Area* of the □ QqpP; wherefore (by *Sch.* 4.) $a \times \overline{\overline{x - a}|^m + \overline{x - 2a}|^m + \overline{x - 3a}|^m + \overline{x - 4a}|^m}$ + &c. to $\frac{y}{a}$ Places continu'd = $a \times x \times \frac{y}{a} - a^{m+1} \times \frac{\overline{\frac{y}{a}}|^{m+1}}{m+1} + \frac{1}{2} \times \overline{\frac{y}{a}}|^m \phi$ (by Part 13 or 14, or by 1ſt *Coroll.* of our *Lemma* 2.) $= xy - \frac{y^{m+1}}{m+1} - \frac{y^m a}{2} \phi = \frac{myx}{m+1} - \frac{1}{2} xa\phi$ (ſince y^m is equal x) is = the *Sum* of the *Area's* of the inſcribed □ *s*, which is therefore ⊐ *Area* of the *Semi-parabola* ABC:

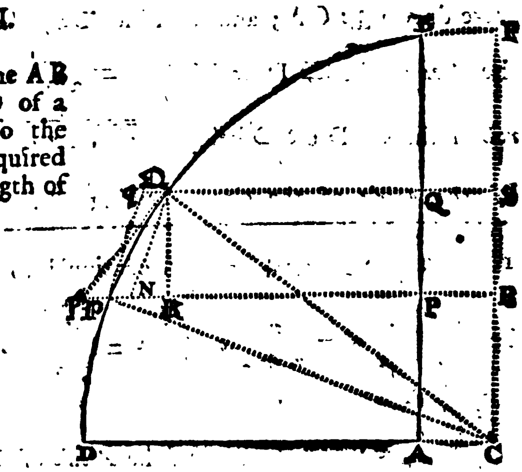
And $a \times \overline{\overline{x - 0} + \overline{x - a}|^m + \overline{x - 2a}|^m + \overline{x - 3a}|^m}$ + &c. to $\frac{y}{a}$ Places continu'd, is manifeſtly = the Sum of the inſcrib'd □s $+ a^{m+1} \times \overline{\frac{y}{a}}|^m = \frac{myx}{m+1} + \frac{1}{2} xa\phi$ (ſince y^m is equal to x) is = the Sum of the *Area's* of the circumſcribing □ *s*, which is therefore ⊏ *Area* of the *Semi-parabola* ABC; conſequently (by our *Lemma* 1.) $\frac{m}{m+1} yx$ is = *Area* of the *Semi-parabola* ABC.

Scholium.

If the Line CA had been ſuppos'd to be divided into an indefinite Number of equal Parts, each Part being = *a*, and thro' any two Points D and E, of thoſe Diviſions, at the Diſtances $\overline{z - 1} \times a$, and za from C, the Lines Dd, Ee were drawn ⊥s CA; and the Lines 𝔇𝔈, *de* ||s CA; then the *Area* of the ▭ DE*ed* being $= \overline{za}|^{\frac{1}{m}} \times a$; and the *Area* of the ▭ DE𝔈𝔇 being $= \overline{z-1 \times a}|^{\frac{1}{m}} \times a$; the Sum of the *Area's* of the circumſcribing ▭*s* $= a^{\frac{1}{m}} \times a \times \overline{1^{\frac{1}{m}} + 2^{\frac{1}{m}} + 3^{\frac{1}{m}} + 4^{\frac{1}{m}} + \text{\&c.}}$ continu'd to $\frac{x}{a}$ Terms muſt be $= \frac{mxy}{m+1} \circ = \frac{m}{m+1} x^{\frac{m+1}{m}} \circ = \frac{x^{\frac{1}{m}+1}}{\frac{1}{m}+1} \circ = \overline{a}|^{\frac{1}{m}+1} \times \frac{\overline{\frac{x}{a}}|^{\frac{1}{m}+1}}{\frac{1}{m}+1} \circ$: And the Sum of the *Area's* of the inſcrib'd ▭*s* $= a^{\frac{1}{m}} \times a \times \overline{0 + 1^{\frac{1}{m}} + 2^{\frac{1}{m}} + 3^{\frac{1}{m}} + 4^{\frac{1}{m}} + \text{\&c.}}$ continu'd to $\frac{x}{a}$ Terms muſt be $= \overline{a}|^{\frac{1}{m}+1} \times \frac{\overline{\frac{x}{a}}|^{\frac{1}{m}+1}}{\frac{1}{m}+1} \circ$: Whence it is manifeſt that $1^{\frac{1}{p}} + 2^{\frac{1}{p}} + 3^{\frac{1}{p}} + \text{\&c.}$ continu'd to $\begin{Bmatrix} n \\ n-1 \end{Bmatrix}$ Terms (n being indefinite, and $\frac{1}{p} =$ any affirmative Fraction whoſe Numerator is 1) is $= \frac{n^{\frac{1}{p}+1}}{\frac{1}{p}+1} \begin{Bmatrix} \circ \\ \circ \end{Bmatrix}$ [*See our Lem.* 2.]

Problem III.

Given the Sine AB of the Arc BD of a Circle; as alſo the *Radius* DC required to find the Length of the Arc DB.



$$\left.\begin{array}{l} BA = s \\ DC = r \end{array}\right\} \text{Given.}$$

DB = [illegible] required.

Suppoſe the Line (or Sine) AB to be divided into an indefinite Number of equal Parts, and each Part ($= PQ$) $= a$: Thro' any two Points Q and P, of which Diviſions at the Diſtances $zA = AQ$, and $\overline{z-1} \times a = AP$ from A, draw the Lines $SQ\mathfrak{D}q$ and $RP\mathfrak{K}N\mathfrak{P}p \perp s$ AB: Draw the Line $\mathfrak{D}p$ ſo as to touch the Arc in $\mathfrak{D}$, and interſect Rp in p; draw alſo the Line $\mathfrak{P}q$ ſo as to touch the Arc in $\mathfrak{P}$; and draw $N\mathfrak{D} \parallel \mathfrak{P}q$: Draw the Lines CF and $\mathfrak{K}\mathfrak{D} \parallel s$ AB; and laſtly, draw the Lines $\mathfrak{D}C$ and $\mathfrak{P}C$. Then 'tis plain, that the indefinitely-ſhort circumſcribing Line $\mathfrak{D}p$ is greater than the Tangent of its reſpective Arc, and conſequently greater, or longer than $\mathfrak{D}\mathfrak{P}$; and that their reſpective indefinitely-ſhort inſcrib'd Line $\mathfrak{D}N$ is leſs than the Sine of the Arc $\mathfrak{D}\mathfrak{P}$, and conſequently leſs or ſhorter than $\mathfrak{D}\mathfrak{P}$.

Solution.

The $\llcorner d\,\triangle s$ $C\mathfrak{D}S$ and $p\mathfrak{D}\mathfrak{K}$ are ſimilar; for the $\angle p\mathfrak{D}\mathfrak{K}$ = Complement of $\angle C\mathfrak{D}R$ to a $\llcorner$ is $= \angle C\mathfrak{D}S$; conſequently $\sqrt{rr - z^2a^2}$ $(\mathfrak{D}S) : r$ $(C\mathfrak{D}) :: a$ $(\mathfrak{D}\mathfrak{K}) :$ $\frac{ra}{\sqrt{rr - z^2a^2}} = p\mathfrak{D}$.

Again, the ⊾d △s $N\mathfrak{Q}\mathfrak{n}$ and $R\mathfrak{P}C$ are ſimilar; for the $\angle\mathfrak{Q}N\mathfrak{n} = \angle\mathfrak{n}\mathfrak{P}\mathfrak{p}$ = Complement of $\angle R\mathfrak{P}C$ to a ⊾ is $= \angle\mathfrak{P}CR$; wherefore $\sqrt{rr - \overline{z-1}|^2a^2}\ (\mathfrak{P}R) : r\ (\mathfrak{P}C) :: a\ (\mathfrak{Q}R) : \frac{ra}{\sqrt{r^2 - \overline{z-1}|^2a^2}} = \mathfrak{Q}N$; wherefore (by *Schol.* 4.) $ar \times \overline{\overline{rr-aa}|^{-\frac{1}{2}} + \overline{rr-2^2a^2}|^{-\frac{1}{2}} + \overline{rr-3^2a^2}|^{-\frac{1}{2}}}$ + &c. to $\frac{s}{a}$ Places continu'd is = the Sum of the circumſcribing Lines, and therefore $\sqsubset \mathfrak{A}$.

And $ra \times \overline{\overline{rr-0}|^{-\frac{1}{2}} + \overline{rr-a^2}|^{-\frac{1}{2}} + \overline{rr-2^2a^2}|^{-\frac{1}{2}}}$ + &c. to $\frac{s}{a}$ Places continu'd is = the Sum of the inſcrib'd Lines, and is therefore $\sqsupset \mathfrak{A}$: Whence the Sum of the circumſcribing Lines exceeds that of the inſcrib'd Lines only by the indefinitely-little Quantity $ar \times \overline{\overline{r^2-s^2}|^{-\frac{1}{2}} - \overline{rr-0}|^{-\frac{1}{2}}}$ wherefore the Sum of either Series is indefinitely-near $= \mathfrak{A}$. But the Sum of either Series cannot be had in finite Terms; wherefore we muſt have Recourſe to Approximations by an infinite Series; thus, $\mathfrak{Q}\mathfrak{p} = ar \times \overline{rr - z^2a^2}|^{-\frac{1}{2}}$ is (by Sir *Iſaac Newton*'s Theorem)

$$= ar \times \overline{\overline{rr}|^{-\frac{1}{2}}\left(\frac{1}{r}\right) + \frac{z^2a^2}{2r^3} + \frac{3 \times z^4a^4}{8r^5} + \frac{15 \times z^6a^6}{48r^7} + \&c.}$$

continu'd *in Infinitum.*

Wherefore (by *Schol.* 4.)

$$ar \times \overline{\frac{1}{r} + \frac{a^2}{2r^3} + \frac{3a^4}{8r^5} + \frac{15a^6}{48r^7} + \&c.}\ \textit{In Infinitum.}$$

$$+ ar \times \overline{\frac{1}{r} + \frac{2^2a^2}{2r^3} + \frac{3 \times 2^4a^4}{8r^5} + \frac{15 \times 2^6a^6}{48r^7} + \&c.}\ \textit{In Infinitum.}$$

$$+ ar \times \overline{\frac{1}{r} + \frac{3^2a^2}{2r^3} + \frac{3 \times 3^4a^4}{8r^5} + \frac{15 \times 3^6a^6}{48r^7} + \&c.}\ \textit{In Infinitum.}$$

+ &c. to $\frac{s}{a}$ Series continu'd.

$$= s + \frac{s^3}{6r^2} + \frac{s^2a}{4r^2}\phi + \frac{3s^5}{40r^4} + \frac{3s^4a}{16r^4}\phi + \frac{15s^7}{336r^6} + \frac{15s^6a}{96r^6}\phi$$

+ &c. *In Infinitum* (by adding the firſt Members of each Series into one Sum; alſo the ſecond Members into another; alſo the third Members into another, &c. by our *Lem.* 2.) is = the Sum of the circumſcribing Lines, and therefore ⊏ $\mathfrak{A}$. And the Sum of the circumſcribing Lines $= ar \times \overline{\overline{r^2 - s^2}|^{-\frac{1}{2}} - \overline{rr}|^{-\frac{1}{2}}}$

$$= s + \frac{s^3}{6r^2} - \frac{s^2a}{4r^2}\phi + \frac{3s^5}{40r^4} - \frac{3s^4a}{16r^4}\phi + \frac{15s^7}{336r^6} - \frac{15s^6a}{96r^6}\phi,$$

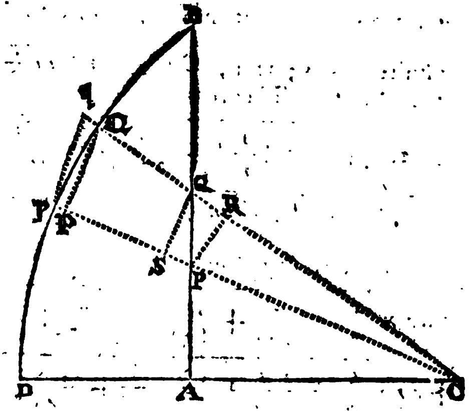
&c. *In Infinitum*, is (by what has been before ſaid) = the Sum of the inſcrib'd Lines, and therefore ⊐ $\mathfrak{A}$.

Conſequently (by our *Lem.* 1.) $s + \frac{s^3}{6r^2} + \frac{3s^5}{40r^4} + \frac{15s^7}{336r^6}$ + &c. *Sine fine* is = $\mathfrak{A}$.

That is to ſay, if you reduce $\overline{rr - ss}|^{-\frac{1}{2}}$ to an infinite Series, and multiply each Member of that Series by sr, and then divide each Member of the Product by the *Index* of s, in that Member you will have the Value of $\mathfrak{A}$ required.

Otherwiſe thus;

Suppoſe the Line (or Sine) AB to be divided into an indefinite Number of equal Parts, and each Part = a (= PQ) and thro' any two Points Q and P of thoſe Diviſions at the Diſtances za = AQ, and $\overline{z - 1} \times a$ = AP from A, draw the Lines CQ 𝔔q, and CP 𝔓p. Draw the Lines qp, 𝔔𝔓, and QS ⊥ Cp; and PR ⊥ Cq: And ſuppoſe AC = $(\sqrt{rr - ss}) = c$.



si enim duceretur recta a C ad B, esset ACq + ABq = CBq = CDq.

Solu-

Solution.

The Δ*s* CAQ and PQR are ſimilar; wherefore $\sqrt{cc+z^2a^2}\,(CQ) \cdot\cdot\, c\,(AC) :: a\,(PQ) \cdot\cdot\, \frac{ca}{\sqrt{cc+zzaa}} = PR.$

And the Δ*s* CPR and C$\mathfrak{p}\mathfrak{Q}$ are ſimilar; wherefore $\sqrt{cc+zzaa}\,(CQ) \cdot\cdot\, \frac{ca}{\sqrt{cc+zzaa}}\,(PR) :: r\,(C\mathfrak{Q}) \cdot\cdot\, \frac{rca}{cc+zzaa}$ which is* $\sqsupset$ inſcrib'd Line (or Sine) $\mathfrak{Q}\mathfrak{p}$.

*Becauſe CQ is $\sqsubset$ CP.

Again, the Δ*s* CPA and SQP are ſimilar; wherefore $\sqrt{cc+\overline{z-1}|^2a^2}\,(CP) \cdot\cdot\, c\,(AC) :: a\,(PQ) \cdot\cdot\, \frac{ca}{\sqrt{cc+\overline{z-1}|^2a^2}} = SQ.$

And the Δ*s* CSQ and C$\mathfrak{p}\mathfrak{q}$ are ſimilar; wherefore $\sqrt{cc+\overline{z-1}|^2a^2}\,(CP) \cdot\cdot\, \frac{ca}{\sqrt{cc+\overline{z-1}|^2a^2}}\,(SQ) :: r\,(CP) \cdot\cdot\, \frac{rca}{cc+\overline{z-1}|^2a^2}$ which is* $\sqsubset$ circumſcribing Line (or Tangent) $\mathfrak{p}\mathfrak{q}$; wherefore (by *Scholium* 4.)

$rca \times \overline{\frac{1}{cc+aa} + \frac{1}{c^2+2^2a^2} + \frac{1}{c^2+3^2a^2} + \&c. \text{ to } \frac{s}{a}}$ places continu'd is $\sqsupset$ Sum of the inſcrib'd Lines, and conſequently $\sqsupset$ $\mathfrak{A}$.

And $rca \times \overline{\frac{1}{c^2} + \frac{1}{c^2+a^2} + \frac{1}{c^2+2^2a^2} + \&c.}$ continu'd to $\frac{s}{a}$ Places is $\sqsubset$ Sum of the circumſcribing Lines, and conſequently $\sqsubset$ $\mathfrak{A}$.

Hence 'tis manifeſt, that the Sum of the laſt (or greater) Series exceeds that of the firſt (or leſſer) Series, only by the indefinitely-little Quantity $rca \times \overline{\frac{1}{cc} - \frac{1}{cc+ss}}$; wherefore the Sum of either Series is indefinitely near $=$ $\mathfrak{A}$: But the Sum of either Series cannot be had in finite Terms; wherefore we muſt have Recourſe to Approximations; thus, $\frac{rca}{cc + zzaa}$ is (by Diviſion) $= rca \times \overline{\frac{1}{cc} - \frac{z^2a^2}{c^4} + \frac{z^4a^4}{c^6} - \frac{z^6a^6}{c^8}}$, &c. *In Infinitum.* wherefore (by *Scholium* 4.)

$$rca \times \overline{\frac{1}{cc} - \frac{aa}{c^4} + \frac{a^4}{c^6} - \frac{a^6}{c^8}}, \text{ \&c. } \textit{In Infinitum.}$$

$$+ rca \times \overline{\frac{1}{cc} - \frac{2^2a^2}{c^4} + \frac{2^4a^4}{c^6} - \frac{2^6a^6}{c^8}}, \text{ \&c. } \textit{In Infinitum.}$$

$$+ rca \times \overline{\frac{1}{cc} - \frac{3^2a^2}{c^4} + \frac{3^4a^4}{c^6} - \frac{3^6a^6}{c^8}}, \text{ \&c. } \textit{In Infinitum.}$$

$+$ &c. to $\frac{s}{\phi}$ Series continu'd;

$$= \frac{rs}{c} - \frac{rs^3}{3c^3} - \frac{rs^2a}{2c^3}\phi + \frac{rs^5}{5c^5} + \frac{rs^4a}{2c^5}\phi - \frac{rs^7}{7c^7} - \frac{rs^6a}{2c^7}\phi,$$

&c. *In Infinitum* (by our *Lem.* 2.) is $\sqsupset$ Sum of the inſcrib'd Lines (or Sines) and conſequently $\sqsupset \mathfrak{A}$.

But the Sum of the foregoing Series $+$ the indefinitely-little Quantity $rca \times \overline{\frac{1}{cc} - \frac{1}{cc + ss}}$

$$= \frac{rs}{c} - \frac{rs^3}{3c^3} + \frac{rs^2a}{2c^3}\phi + \frac{rs^5}{5c^5} - \frac{rs^4a}{2c^5}\phi - \frac{rs^7}{7c^7} + \frac{rs^6a}{2c^7}\phi, \text{ \&c.}$$

is $\sqsubset$ Sum of the circumſcribing Lines (or Tangents;) and therefore $\sqsubset \mathfrak{A}$.

Now, ſince the Sum of the latter Series is $\sqsubset$ Sum of the former $\frac{r^2a}{2c^3} - \frac{rs^4a}{2c^5} + \frac{rs^6a}{2c^7}\phi$, &c. *In Infinitum*, is affirmative (but withal indefinitely-little) for it is $\sqsubset - \frac{r^2a}{2c^3} + \frac{rs^4a}{2c^5} - \frac{rs^6a}{2c^7}\phi$, &c. *In Infinitum*, which is therefore Negative; conſequently (by our *Lem.* 1.)

$\frac{rs}{c} - \frac{rs^3}{3c^3} + \frac{rs^5}{5c^5} - \frac{rs^7}{7c^7}$, &c. *Sine fine*, is $= \mathfrak{A}$.

That is to ſay, if you reduce $\frac{rsc}{cc + ss}$ to an infinite Series, (which is done by Diviſion) and then divide each Member in that Series by the *Index* of s, in the ſaid Member you will have the Value of z required.

Scholium.

By either of theſe two Solutions, the Proportion of the Circumference of a Circle to its Diameter may be found, and expreſs'd in Numbers very near the Truth; thus, by the laſt Solution.

Suppoſe the Arc BD = 30°, and r (= *Radius*) = 1; then s will be = .5, and $c = \sqrt{.75}$; and 6 × BD will be

$$= 180^\circ = 6 \times \overline{\frac{.5}{\sqrt{.75}} - \frac{.5^3}{3\sqrt{.75}} + \frac{.5^5}{5\sqrt{.75}}, \&c.} =$$

$$6 \times \frac{.5}{\sqrt{.75}} (= \sqrt{12}) \times \overline{1 - \frac{1}{3 \times 3} + \frac{1^2}{5 \times 3^2} - \frac{1^3}{7 \times 3^3}, \&c.}$$

$$= \sqrt{12} \times \overline{1 - \frac{1}{3 \times 3} + \frac{1}{5 \times 3^2} - \frac{1}{7 \times 3^3} + \frac{1}{9 \times 3^4}, \&c.}$$

wherefore $1 \cdot\cdot \sqrt{12} \times \overline{1 - \frac{1}{3 \times 3} + \frac{1}{5 \times 3^2}, \&c.}$ :: *Radius* ·· 180° :: Diameter ·· Circumference of any Circle.

10
16
1400
1116
128400
27696
81|170400
69301
820|10190000

Operation.

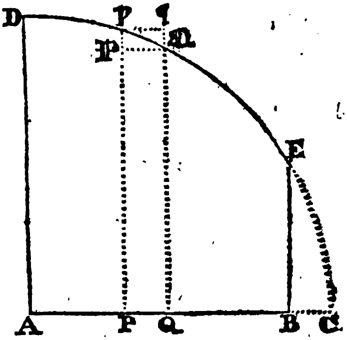
$\sqrt{12} \times 1 = 3.464101615$	$\sqrt{12} \times 1 = 3.464101615$
$\sqrt{12} \times \frac{1}{3^2} = .384900179$	$\sqrt{12} \times \frac{1}{5 \times 3^2} = 76980035$
$\sqrt{12} \times \frac{1}{3^4} = 42766686$	$\sqrt{12} \times \frac{1}{9 \times 3^4} = 4751854$
$\sqrt{12} \times \frac{1}{3^6} = 4751854$	$\sqrt{12} \times \frac{1}{13 \times 3^6} = 365527$
$\sqrt{12} \times \frac{1}{3^8} = 527983$	$\sqrt{12} \times \frac{1}{17 \times 3^8} = 31057$
$\sqrt{12} \times \frac{1}{3^{10}} = 58664$	$\sqrt{12} \times \frac{1}{21 \times 3^{10}} = 2793$
$\sqrt{12} \times \frac{1}{3^{12}} = 6518$	$\sqrt{12} \times \frac{1}{25 \times 3^{12}} = 260$
$\sqrt{11} \times \frac{1}{3^{14}} = 724$	$\sqrt{12} \times \frac{1}{29 \times 3^{14}} = 24$
$\sqrt{12} \times \frac{1}{3^{16}} \left(= \sqrt{12} \times \frac{1}{3^{14}} \times \frac{1}{9} \right) = 80$	$\sqrt{12} \times \frac{1}{33 \times 3^{16}} = 2$
	3.546233167

$\sqrt{12} \times - \frac{1}{3} = -1.154700538$	$\sqrt{12} \times - \frac{1}{3 \times 3} = -.384900179$
$\sqrt{12} \times - \frac{1}{3^3} = -.128300059$	$\sqrt{12} \times - \frac{1}{7 \times 3^3} = -18328579$
$\sqrt{12} \times - \frac{1}{3^5} = - 14255562$	$\sqrt{12} \times - \frac{1}{11 \times 3^5} = - 1295960$
$\sqrt{12} \times - \frac{1}{3^7} = - 1583951$	$\sqrt{12} \times - \frac{1}{15 \times 3^7} = - 105596$
$\sqrt{12} \times - \frac{1}{3^9} = - 175994$	$\sqrt{12} \times - \frac{1}{19 \times 3^9} = - 9262$
$\sqrt{12} \times - \frac{1}{3^{11}} = - 19554$	$\sqrt{12} \times \frac{1}{23 \times 3^{11}} = - 850$
$\sqrt{12} \times - \frac{1}{3^{13}} = - 2172$	$\sqrt{12} \times \frac{1}{27 \times 3^{13}} = - 80$
$\sqrt{12} \times - \frac{1}{3^{15}} = - 241$	$\sqrt{12} \times \frac{1}{31 \times 3^{15}} = - 7$
	$-.404640513$

$$\begin{array}{r} 3.546233167 \\ -\ .404640513 \\ \hline 3.14159265 \end{array}$$

That is, the Diameter of any Circle is to its Circumference as 1 to 3.14159265 very near.

Problem. IV.



Given the *Radius* AD of the Quadrant ACD; as alſo the Right Line AB; and conſequently BE ⊥ AB: Required to find the *Area* of the Part ABED of the ſaid Quadrant.

$$\left.\begin{array}{r}\text{AD}\,(=\text{AC}) = r\\ \text{AB} = y\end{array}\right\}\text{Given.}$$

Area of the Space ABED $= \mathfrak{A}$ required.

Suppoſe the Line AB to be divided into an indefinite Number of equal Parts; and each Part $= a$ $(=$ PQ$)$: And thro' any two Points of thoſe Diviſions as Q and P, at the Diſtances zA = AQ, and $\overline{z-1}\times a =$ AP from A, draw the Lines Q𝔔q and P𝔓𝔭 ⊥*s* AB: And draw the Lines 𝔓𝔔 and 𝔭𝔮 ||*s* AB.

Solution.

By the Property of a Circle $\sqrt{\overline{r+za}\times\overline{r-za}}$ $(=\sqrt{\overline{r+\text{AQ}}\times\overline{\text{QC}}} = \sqrt{r^2 - z^2a^2} =$ Q𝔔: And $\sqrt{\overline{r+\overline{z-1}\times a}\times\overline{r-\overline{z-1}\times a}} = (\sqrt{\overline{r+\text{AP}}\times\text{PC}})$ $= \sqrt{r^2 - \overline{z-1}|^2a^2} =$ P𝔭; wherefore $a\times\sqrt{r^2 - z^2a^2}$ is $=$ *Area* of the inſcrib'd ▭ PQ𝔔𝔓:

And $a\sqrt{r^2 - \overline{z-1}|^2a^2}$ is $=$ *Area* of the circumſcribing ▭ PQ𝔮𝔭; wherefore (by *Schol.* 4.)

$a\times\overline{\sqrt{r^2-a^2}+\sqrt{r^2-2^2a^2}+\sqrt{r^2-3^2a^2}+\text{\&c. to}}$ $\overline{\frac{y}{a}\text{ Places continu'd}}$ is = the Sum of the *Area's* of the inſcribed ▭*s*, and therefore ⊐ $\mathfrak{A}$.

And $a\times\overline{\sqrt{rr}+\sqrt{r^2-a^2}+\sqrt{r^2-2^2a^2}+\text{\&c. to}}$ $\overline{\frac{y}{a}}$ Places continu'd is = the Sum of the *Area's* of the circumſcribing ▭*s*, and therefore ⊏ $\mathfrak{A}$.

Hence 'tis manifeſt, that the Sum of the circumſcribing ▭*s* exceeds that of the inſcrib'd ▭*s* only by the indefinitely-little Quantity $ar - a\sqrt{rr - yy}$; conſequently the Sum of either Series is indefinitely-near $= \mathfrak{A}$: But the Sum of either Series cannot be had in finite Terms; wherefore we muſt have Recourſe to Approximations; thus,

$a \times \sqrt{rr - zzaa}$ is (by Sir *Iſaac Newton*'s Theorem)

$$= a \times \overline{r - \frac{z^2a^2}{2r} - \frac{z^4a^4}{8r^3} - \frac{z^6a^6}{16r^5} - \&c.\ \textit{Sine fine}};$$

Wherefore (by *Schol.* 4.)

$$a \times \overline{r - \frac{a^2}{2r} - \frac{a^4}{8r^3} - \frac{a^6}{16r^5} - \&c.\ \textit{In Infinitum.}}$$

$$+ a \times \overline{r - \frac{2^2a^2}{2r} - \frac{2^4a^4}{8r^3} - \frac{2^6a^6}{16r^5} - \&c.\ \textit{In Infinitum.}}$$

$$+ a \times \overline{r - \frac{3^2a^2}{2r} - \frac{3^4a^4}{8r^3} - \frac{3^6a^6}{16r^5} - \&c.\ \textit{In Infinitum.}}$$

$+$ &c. to $\frac{y}{a}$ Series continu'd.

$$= ry - \frac{y^3}{6r} \circ - \frac{y^5}{40r^3} \circ - \frac{y^7}{112r^5} \circ - \&c.\ \textit{In Infinitum},$$

(by our *Lemma* 2.) is $=$ the Sum of the *Area's* of the inſcrib'd ▭*s*, and therefore $\sqsupset \mathfrak{A}$.

And the Sum of the foregoing Series $+ ar - a\sqrt{rr - yy}$

$$= ry - \frac{y^3}{6r} \circ - \frac{y^5}{40r^3} \circ - \frac{y^7}{112r^5} \circ, \&c.\ \textit{In Infinitum},$$ is

$=$ the Sum of the *Area's* of the circumſcribing ▭*s*, and therefore $\sqsubset \mathfrak{A}$: Conſequently (by our *Lemma* 1.) ry

$$- \frac{y^3}{6r} - \frac{y^5}{40r^3} - \frac{y^7}{112r^5} - \frac{5y^9}{1152r^7} - \&c.\ \textit{Sine fine} \text{ is } = \mathfrak{A}.$$

That is to ſay, if you reduce $\sqrt{rr - yy}$ to an endleſs Series, and multiply each Member of that Series by y, and then divide each Member of the Product by the *Index* of y, in that Member you will have the Value of $\mathfrak{A}$ required.

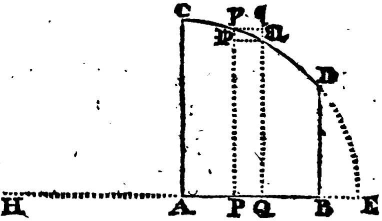
Corol-

Corollary.

Hence the *Area* of the *Quadrant* ADC is (becauſe y, in this Caſe, becomes $=$ to r) $= r^2 - \frac{r^2}{6} - \frac{r^2}{40} - \frac{r^2}{112} - \frac{5r^2}{1152} -$ &c. *Sine fine.*

Note, *By the help of this* Problem 4, *the Proportion of the Circumference of a Circle to its Diameter may be found very near.*

Problem V.



Given the Semi-conjugate Diameter AC, as alſo the Part AB of the Given Semi-tranſverſe Diameter AE; and conſequently BD ∥ AC required to find the *Area* of the Portion ABDC of an *Ellipſis.*

$$\left.\begin{aligned} AE &= r \\ AB &= y \\ AC &= c \end{aligned}\right\}\text{Given.}$$

Area of the Part ACDB of the $\frac{1}{4}$ *Ellipſe* ACE $= \mathfrak{A}$ required.

Suppoſe the Line AB to be divided into an indefinite Number of equal Parts, and each part $= a =$ PQ; and at the indetermin'd Diſtances $za =$ AQ; and $\overline{z-1} \times a =$ AP from A, erect the Lines Q$\mathfrak{Q}$q, and P$\mathfrak{P}$p ⊥'s AB, and draw the Lines qp and $\mathfrak{Q}\mathfrak{P}$ ∥s AB:

Solution.

$$\text{By the Property of an } \textit{Ellipſe.}\left\{\begin{aligned} &r\,(EA) \,..\, c\,(AC) :: \overline{\overline{r+z}\,a \times \overline{r-z}\,a}\big|^{\frac{1}{2}} \\ &(\overline{HQ \times QE}\big|^{\frac{1}{2}}) \,..\, \frac{c}{r} \times \overline{rr - zzaa}\big|^{\frac{1}{2}} \\ &= Q\mathfrak{Q} :: \overline{r + \overline{z-1} \times a \times r - \overline{z-1} \times a}\big|^{\frac{1}{2}} \\ &(\sqrt{HP \times PE}) \,..\, \frac{c}{r} \times \overline{rr - \overline{\overline{z-1} \times a}\big|^{2}}\big|^{\frac{1}{2}} = P\mathfrak{p} \end{aligned}\right.$$

Wherefore $\frac{ca}{r} \times \overline{rr - za|^{2}}|^{\frac{1}{2}}$ is = *Area* of ▭ PQ𝔔𝔓:

And $\frac{ca}{r} \times \overline{rr - \overline{z - 1 \times a}|^{2}}|^{\frac{1}{2}}$ is = *Area* of ▭ PQ𝔮𝔭; wherefore it is plain from the next preceding *Problem*, that

$$\frac{c}{r} \times \overline{ry - \frac{y^{3}}{6r} - \frac{y^{5}}{40r^{3}} - \frac{y^{7}}{112r^{5}} - \frac{5y^{9}}{1152r^{7}} + \frac{7y^{11}}{2816r^{9}} - \&c.}$$

$\overline{\textit{Sine fine}}$, is = 𝔄.

Scholium.

Hence the *Area* of $\frac{1}{4}$ the *Ellipſe*; that is, the *Area* of ACE is (becauſe y, in this Caſe, becomes $= r$)

$$= \frac{c}{r} \times \overline{rr - \frac{r^{2}}{6} - \frac{r^{2}}{40} - \frac{r^{2}}{112} - \frac{5r^{2}}{1152} - \&c.} \textit{ Sine fine.}$$

Corollary.

But $2c \cdot\cdot\ 2r :: \frac{c}{r} \times \overline{rr - \frac{r^{2}}{6} - \frac{r^{2}}{40} - \frac{r^{2}}{112} - \frac{5r^{2}}{1152} - \&c.}$

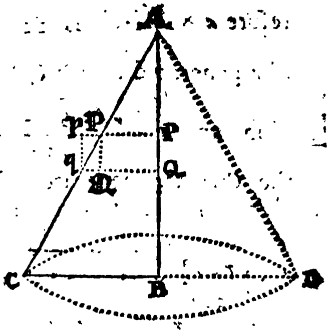
$\overline{\textit{Sine fine}}, \cdot\cdot\ rr - \frac{r^{2}}{6} - \frac{r^{2}}{40} - \frac{r^{2}}{112} - \frac{5r^{2}}{1152} - \&c.$ *ad infinitum*;

That is, the conjugate Diameter is to the Tranſverſe Diameter, as the *Area* of $\frac{1}{4}$ the *Ellipſe* is to the * *Area* of $\frac{1}{4}$ the Circle, deſcrib'd by the ſaid Tranſverſe Diameter as a *Radius*, that is, as the *Area* of the whole *Ellipſe* is to the *Area* of the whole Circle deſcrib'd on the *Radius* AE.

* *See the* Corol. *to* Prob. 4.

Problem

Problem VI.



Given the *Axis* A B of the *Cone* A C B D form'd by the Revolution of the ⊾*d* △ ABC, upon the ſaid *Axis*, as alſo the *Radius* BC of its Baſe, and * conſequently the Circumference deſcrib'd by the Point C in the ſaid Revolution. Required to find the Solidity of the ſaid *Cone*. [* *See* Prob. 3.]

$$\left.\begin{array}{r}\text{AB} = b\\ \text{CB} = \text{BD} = r\\ \text{Circumference of the Baſe} = c\end{array}\right\}\text{Given.}$$

Solidity of the *Cone* A B C D $= \mathfrak{S}$ required.

Suppoſe the *Axis* A B to be divided into an indefinite Number of equal Parts, and each of thoſe Parts to be $= a =$ PQ, and thro' any two of thoſe Diviſions as Q and P, at the indetermin'd Diſtances $za =$ AQ and $\overline{z-1}\times a =$ AP from A, draw the Lines Q$\mathfrak{Q}$$\mathfrak{q}$, and P$\mathfrak{P}$$\mathfrak{p}$ ⊥*s* AB; and draw the Lines $\mathfrak{q}\mathfrak{p}$ and $\mathfrak{Q}\mathfrak{P}$ ||*s* AB.

Solution.

The △*s* ABC, AQ$\mathfrak{q}$, and AP$\mathfrak{P}$ are ſimilar; wherefore

$$b\,(\text{AB}) \,..\, r\,(\text{BC}) :: \left\{\begin{array}{l} za\,(\text{AQ}) \,..\, \dfrac{zar}{b} = \text{Q}\mathfrak{q}.\\[2ex] \overline{z-1}\times a\,(\text{AP}) \,..\, \dfrac{\overline{z-1}\times ar}{b} = \text{P}\mathfrak{P}.\end{array}\right.$$

And ſince * Circles are in Proportion to one another as the Squares of their *Radii*, r^2 (CBq:) .. $\frac{1}{2}rc$ (*Area* of the Circle deſcrib'd by the *Radius* CB in the ſaid Revolution) :: $\frac{z^2a^2r^2}{bb}$ ($\overline{\text{Q}\mathfrak{q}}|^2$) .. $\frac{rcz^2a^2}{2b^2}$ (*Area* of the Circle deſcrib'd by the Line Q$\mathfrak{q}$ in the ſaid Revolution) :: $\frac{\overline{\overline{z-1}\times a}|^2\,r^2}{b^2}$ (P$\mathfrak{P}$$q$:) .. $\frac{rc\times\overline{\overline{z-1}\times a}|^2}{2bb} =$ *Area* of the Circle deſcrib'd by the Line P$\mathfrak{P}$ in the ſaid Revolution; where-

* i. e. *their* Area's.

wherefore $a \times \frac{rcz^2a^2}{2b^2}$ is = * Solidity of the *Cylinder* generated by ▭ QPpq in the ſaid Revolution, and $a \times \frac{rc \times \overline{z-1 \times a}|^2}{2b^2}$ is = Solidity of the *Cylinder* generated by ▭ QP𝔓𝔔 in the aforeſaid Revolution; wherefore (by our

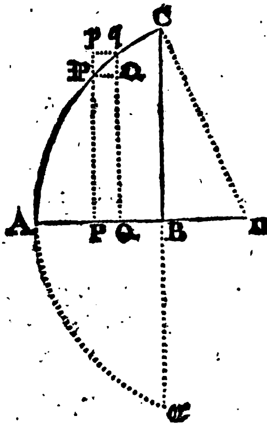
Note, I ſuppoſe it known, that the Solidity of a Cylinder *is* = Area *of its Baſe* × *its Height.*

Schol. 4.) $\frac{rca^3}{2bb} \times \overline{1^2+2^2+3^2+4^2+\&c. \text{ to } \frac{b}{a} \text{ Terms continu'd}} = \frac{rcb}{6} + \frac{rca}{4}\phi$, (by our *Lem.* 2.) is = Sum of the Solidities of the indefinitely-little circumſcribing *Cylinders*, and therefore $\sqsubset$ 𝔄:

And $\frac{rca^3}{2b^2} \times \overline{0+1^2+2^2+3^2+\&c. \text{ to } \frac{b}{a} \text{ Terms continu'd}} = \frac{rcb}{6} - \frac{rca}{4}\phi$, is = the Sum of the Solidities of the inſcrib'd indefinitely-little *Cylinders*, and therefore $\sqsupset$ 𝔄; conſequently (by our *Lem.* 1.) $\frac{rcb}{6}$ is = 𝔄.

Problem VII.

Given the Line AB in the Segment CAℭ of a Sphere, (the ſaid Segment being formed by the Revolution of the Semi-ſegment BAC, upon its *Axis* AD) as alſo the *Radius* AD, and conſequently the Circumference of the greateſt Circle of the ſaid Sphere: Required to find the Solidity of the ſaid Segment.



$$\left.\begin{array}{r} AB = b \\ AD = r \\ \text{Circumference of a great Circle of the } Sphere = c \end{array}\right\} \text{Given.}$$

$$\left.\begin{array}{l} \text{Solidity of the Segment CAℭ of the Sphere (gene-} \\ \text{rated as aforeſaid)} = 𝔄 \end{array}\right\} \text{ſought.}$$

Preparation.

Suppoſe the Line A B to be divided into an indefinite Number of equal Parts, and each Part $= a =$ P Q, and thro' any two Points Q, P of thoſe Diviſions, at the Diſtances $za = $ A Q, and $\overline{z-1}\times a = $ A P from A, draw the Lines Q $\mathfrak{Q}$ q, and P $\mathfrak{P}$ p ⊥*s* A B, and draw the Lines q p and $\mathfrak{Q}\,\mathfrak{P}$ ∥*s* B A.

Solution.

By the Property of a Circle $\overline{\text{Q}q}|^2$ is $= \overline{2r-za}\times za\ (= \overline{2r-\text{AQ}}\times \text{AQ}) = 2rza - z^2a^2$: And $\overline{\text{P}\mathfrak{P}}|^2$ is $= \overline{2r-\overline{z-1}\times a}\times\overline{z-1}\times a\ (= \overline{2r-\text{AP}}\times\text{AP}) = 2r\times\overline{z-1}\times a - \overline{\overline{z-1}\times a}|^2$.

And ſince Circles are to one another in duplicate Proportion of their *Radii*, therefore rr (A D q:) ∙∙ $\frac{1}{2}rc$ (*Area* of one of the great Circles of the Sphere) :: $2rza - z^2a^2$ ($\overline{\text{Q}q}|^2$) ∙∙ $cza - \frac{cz^2a^2}{2r} =$ *Area* of the Circle deſcrib'd by the Line Q q in revolving as aforeſaid :: $2r\times\overline{z-1}\times a - \overline{\overline{z-1}\times a}|^2$ (P $\mathfrak{P}$ q:) ∙∙ $c\times\overline{z-1}\times a - \frac{c\times\overline{z-1}|^2a^2}{2r} =$ *Area* of the Circle deſcrib'd by P $\mathfrak{P}$ in the ſaid Revolution; wherefore $a\times\overline{cza - \frac{cz^2a^2}{2r}}$ is $=$ Solidity of the indefinitely-little *Cylinder* form'd by ▭ Q P p q in the ſaid Revolution: And $a\times\overline{c\times\overline{z-1}\times a - c\times\frac{\overline{z-1}|^2a^2}{2r}}$ is $=$ Solidity of the indefinitely-little *Cylinder* form'd by ▭ Q P $\mathfrak{P}$ $\mathfrak{Q}$ in the ſaid Revolution; wherefore (by our *Schol.* 4.)

$$ac\times\overline{\overline{a-\frac{a^2}{2r}}+\overline{2a-\frac{2^2a^2}{2r}}+\overline{3a-\frac{3^2a^2}{2r}}+\overline{4a-\frac{4^2a^2}{2r}}+\text{\&c.}}$$

to $\frac{b}{a}$ Places continu'd (the two Members under each Line of Conjunction being what is here call'd a Place) $= \overline{\frac{b^2c}{2}+\frac{bca}{2}} - \overline{\frac{b^3c}{6r}+\frac{b^2ca}{4r}}$ (by our *Lem.* 2.) $= \frac{b^2c}{2}-\frac{b^3c}{6r}$

$\bullet$ is = the Sum of the Solidities of the circumſcribing indefinitely-little *Cylinders*, and therefore $\sqsubset \mathfrak{A}$.

And $ac \times \overline{o - o + a - \frac{aa}{2r} + 2a - \frac{2^2a^2}{2r} + 3a - \frac{3^2a^2}{2r} + \&c.}$ to $\frac{b}{a}$ Places continu'd $= \frac{b^2c}{2} - \frac{bca}{2} - \frac{b^3c}{6r} + \frac{b^2ca}{4r} \bullet = \frac{b^2c}{2} - \frac{b^3c}{6r} \bullet$ is = the Sum of the Solidities of the indefinitely-little inſcrib'd *Cylinders*, and therefore $\sqsupset \mathfrak{A}$; whence (by our *Lem.* 1.) $\frac{b^2c}{2} - \frac{b^3c}{6r}$ is = $\mathfrak{A}$.

Scholia.

1. If it were required to find the Solidity of the Figure generated by the Semi-ſector D A C, in revolving upon D A as an *Axis*, it may be briefly done from the two foregoing *Problems*, thus: D B is (B A being equal to b) $= r - b$.

And by the Property of a Circle $\overline{2rb - bb}|^{\frac{1}{2}}$ is = B C: And $r : c :: \overline{2rb - bb}|^{\frac{1}{2}} : \frac{c}{r} \times \overline{2rb - bb}|^{\frac{1}{2}}$; wherefore (by *Problem* 6.)

$$\frac{\overline{r-b} \times \overline{2rb - bb}|^{\frac{1}{2}} \times \frac{c}{r} \times \overline{2rb - bb}|^{\frac{1}{2}}}{6} = \frac{2crrb - 3crb^2 + cb^3}{6r}$$

is = *Area* of the *Cone* form'd by the Revolution of ∟d △ D C B upon D B as an *Axis*; conſequently

$\frac{b^2c}{2} - \frac{b^3c}{6r} + \frac{2cr^2b - 3crb^2 + cb^3}{6r} = \frac{crb}{3}$ is = Solidity of the Solid Figure generated by the Semi-ſector D A C, in revolving upon D A as an *Axis*.

2. Therefore by * *Pardie*'s *Theorem* $\frac{crb}{3} \div \frac{r}{3} = cb$ is = *Area* of the Surface generated by the Arc A C, in revolving upon D A as an *Axis*.

* See Pardie's *Elem.* *Propoſition* — or Sturmius's *Math.* Enucl. *Def.* 20, *conſect.* 1.

3. If b be = [illegible]r, then $2rc$ will be = whole Surface of the Sphere; whence you have the Demonſtration of one of the Great ARCHIMEDES's *Theorems*, namely, that the Surface of a Sphere is quadruple the *Area* of its greateſt Circle; for $2rc$ (the *Area* of the Surface of a Sphere) is $= \frac{1}{2}rc$ (the *Area* of one of its greateſt Circles) $\times 4$.

I

I have apply'd this Method to the finding the *Area's* and *Solidities* of ſeveral other Figures; as alſo to the finding the Nature of *Curves*; among others of that in which a heavy Body ſhall, by the force of its Gravity, deſcend only equal Spaces in equal times; as alſo of the Length of this laſt mention'd *Curve* in reſpect of any given *Abſciſſe*; and of other *Problems* of the like Nature with the latter, not ſolvable by the common Method of Fluxions; that is, without the help of eminential Equations, and after finding the Fluent, putting the fluxionary Letter equal to nothing, *&c.* all which, except the two following *Problems*, I, for Brevity's Sake, omit.

Note, *The Value of any propos'd Figure may be found, by finding the Sum of the Values of either the inſcrib'd or circumſcribing indefinitely-little Figures; for if the former be* $=$ *any Quantity, as ſuppoſe* $s \ominus$, *or the latter* $= s \ominus$, *you may conclude the Value of the propos'd Figure to be* $= s$, *always provided that you inſcribe or circumſcribe the indefinitely-little Figures ſo as ſufficiently to exhauſt, or be exhauſted by the ſaid propos'd Figure.*

Problem VIII.

If the *Abſciſſe* of any Plane Figure be $= x$, and the Proportion of each part of the ſaid *Abſciſſe* to its reſpective Ordinate be as x to $\overline{x+b}\,|^{\frac{1}{n}}$: Then 'tis manifeſt, that the Sum ~~of the~~ *Area's* of the circumſcribing indefinitely-little ▭s will be found

$$= a^{1+\frac{1}{n}} \times \overline{\overline{1+\frac{b}{a}}\,\Big|^{\frac{1}{n}} + \overline{2+\frac{b}{a}}\,\Big|^{\frac{1}{n}} + \overline{3+\frac{b}{a}}\,\Big|^{\frac{1}{n}} + \overline{4+\frac{b}{a}}\,\Big|^{\frac{1}{n}}}$$

$$\overline{+ \text{\&c. to } \frac{x}{a} \text{ Places continu'd}}$$

$$= a^{1+\frac{1}{n}} \times \overline{1^{\frac{1}{n}} + 2^{\frac{1}{n}} + 3^{\frac{1}{n}} + 4^{\frac{1}{n}} + \text{\&c. to } \frac{x+b}{a} \text{ Terms}}$$

$$\overline{\text{continu'd}}$$

$$- a^{1+\frac{1}{n}} \times \overline{1^{\frac{1}{n}} + 2^{\frac{1}{n}} + 3^{\frac{1}{n}} + \text{\&c. to } \frac{b}{a} \text{ Terms continu'd}}$$

$$= a^{1+\frac{1}{n}} \times \overline{\frac{x+b}{a}}\,\Big|^{\frac{1}{n}+1} \bullet \div \overline{\frac{1}{n}+1} - a^{\frac{1}{n}+1} \times \overline{\frac{b}{a}}\,\Big|^{1+\frac{1}{n}} \bullet$$

$$\div \overline{\frac{1}{n}+1} \text{ (by our \textit{Lem.} 2.)} = \frac{n}{n+1} \times \overline{x+b}\,|^{\frac{n+1}{n}} - \frac{n}{n+1}$$

$$\times \overline{b}\,|^{\frac{n+1}{n}} \bullet; \text{ wherefore } \frac{n}{n+1} \times \overline{x+b}\,|^{\frac{n+1}{n}} - \frac{n}{n+1} \times \overline{b}\,|^{\frac{n+1}{n}}$$

is = *Area* of the Figure mention'd in this *Prob.* 8.

Pro-

Problem IX.

If the *Abſciſſe* of any plain Figure be $= x$, and the Proportion of every part of the ſaid *Abſciſſe* to its reſpective Ordinate as x to $x \times \overline{x+b}|^{\frac{1}{n}}$, then the Sum of the *Area's* of the circumſcribing indefinitely-little ▭*s* will be found

$$= a^{2+\frac{1}{n}} \times 1 \times \overline{1+\frac{b}{a}}\Big|^{\frac{1}{n}} + 2 \times \overline{2+\frac{b}{a}}\Big|^{\frac{1}{n}} + 3 \times \overline{3+\frac{b}{a}}\Big|^{\frac{1}{n}}$$

$+$ &c. to $\frac{x}{a}$ Places continu'd

$$= a^{2+\frac{1}{n}} \times \overline{1+\frac{b}{a}}\Big|^{\frac{1}{n}+1} + \overline{2+\frac{b}{a}}\Big|^{\frac{1}{n}+1} + \overline{3+\frac{b}{a}}\Big|^{\frac{1}{n}+1}$$

$+$ &c. to $\frac{x}{a}$ Places continu'd

$$(= a^{2+\frac{1}{n}} \times \overline{1+\frac{b}{a}} \times \overline{1+\frac{b}{a}}\Big|^{\frac{1}{n}} + \overline{2+\frac{b}{a}} \times \overline{2+\frac{b}{a}}\Big|^{\frac{1}{n}} +$$

$$\overline{3+\frac{b}{a}} \times \overline{3+\frac{b}{a}}\Big|^{\frac{1}{n}} + \text{\&c. to } \frac{x}{a} \text{ Places continu'd})$$

$$- a^{2+\frac{1}{n}} \times \frac{b}{a} \times \overline{1+\frac{b}{a}}\Big|^{\frac{1}{n}} + \frac{b}{a} \times \overline{2+\frac{b}{a}}\Big|^{\frac{1}{n}} + \frac{b}{a} \times \overline{3+\frac{b}{a}}\Big|^{\frac{1}{n}}$$

$+$ &c. to $\frac{x}{a}$ Places continu'd

$$= (\text{by } Prob. \ 8.) \ \frac{n}{2n+1} \times \overline{x+b}|^{\frac{2n+1}{n}} \ominus - \frac{n}{2n+1} \times \overline{b}|^{\frac{2n+1}{n}}$$

$$\bullet - \frac{n}{n+1} \times b \times \overline{x+b}|^{\frac{n+1}{n}} \ominus + \frac{n}{n+1} \times b \times \overline{b}|^{\frac{n+1}{n}} \ominus$$

which, after rejecting the Signs $\ominus$ and $\bullet$, is $=$ *Area* of the plain Figure mention'd in this *Prob.* 9.

Theorem.

The Sum of $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, &c. *ſine fine* (upon which depends the Quadrature of that *Hyperbola* whoſe *Abſciſſe* is to its Ordinate as x to x^{-1}) is indefinitely near $=$ *Napier's Logarithm* of $\frac{x}{a}$.

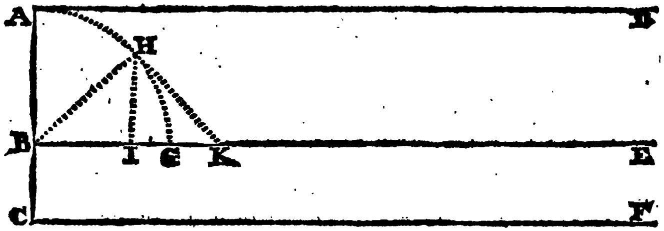
Demon-

Demonſtration.

By *Part* 15, *Chap.* 3. *Rule* 1. the *Logarithm* of $1 - x$ is $= \frac{1000 \text{ \&c. indefinitely}}{n} \times - \overline{x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \text{\&c. } ad\ infin.\ \phi}$.

Wherefore *Napier's Logarithm* of $1 - x$ is (becauſe n in this Caſe is $= 1000$ &c. indefinitely) $= 1 \times - \overline{x + \frac{1}{2}xx + \frac{1}{3}x^3 + \text{\&c. } ad\ infinitum\ \phi}$; wherefore *Napier's Log.* of $\frac{1}{1-x}$ is found (by ſubtracting *Napier's Log.* of $1 - x$ from his *Log.* of 1) $= x + \frac{1}{2}x^2 + \frac{1}{3}xxx + \frac{1}{4}x^4 + \text{\&c. } in\ infinitum\ \phi$; conſequently putting $x = 1$, you have *Napier's Logarithm* of $\frac{1}{0} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{\&c. } in\ infinitum\ \phi$. Q. E. D.

Problem.



Given the Diſtances A B and B C between the three infinite parallel Right Lines A D, B E and C F, bounded on the left Hand at A, B and C, and produc'd infinitely towards the Right: Required to find the Proportion of the *Area* of the infinite Space A B E D, to that of the infinite Space B C F E.

Solution.

On B as a Center, with the *Radius* B A, deſcribe the Circle B G H A, from any Point H of whoſe Circumference let fall H I ⊥ B E: Draw H K ſo as to touch the Circle in H, and interſect B E in K, and draw B H: Then the △s B I H and H I K are ſimilar; wherefore, ſuppoſing the *Radius* B G $= r$, G I $= x$, I H $= y$, and I K $= t$; we have $\frac{yy}{r - x} = t$. But when H I coincides with A B, x, as alſo y, becomes $= r$, and $\frac{yy}{r-x} = t = \frac{rr}{r-r} =$ B E; and conſequently the *Area* of the infinite Space A B E D $= \frac{r^3}{r-r}$. In like manner the *Area* of the infinite Space B C F E will be found (putting B C $= s$) $= \frac{s^3}{s-s}$.

Now